

Atomic weight and infinitesimals in dicotic misère play

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Abstract

The normal play structure of combinatorial games is rich in infinitesimals. Typical examples include the all-small games, also known as dicotic games, which are positions in which one player has an option if and only if the other does. For such games, atomic weight theory and uptimals provide the means of analysis. We present a parallel of this theory for the misère play convention. Firstly, we establish that the uptimals are mirrored by so-called muptimals (a totally ordered set of infinitesimals). Secondly, we show how the normal play atomic weight theory can be applied to certain *tame* dicotic misère rulesets; one can follow the normal play theory until reaching certain well-described endgames, just as one does for misère NIM and other tame impartial rulesets. Finally, we show that this theory breaks down when we consider arbitrary misère structures rather than dicotic ones; in particular, we prove that there are no infinitesimal elements in full misère, and also that the Archimedean property does not hold.

Keywords: Combinatorial game theory, atomic weight, dicotic games, misère play, uptimal notation

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1. Introduction

Combinatorial game theory (CGT) studies alternating games of perfect information. The convention deciding who wins can transform a game’s analysis: under normal play, the winner is the player who makes the last move; in misère play, however, the last to move loses. This simple reversal has made misère theory far more delicate than its normal play counterpart. Against this backdrop, the so-called *dicotic games* will form for us an important class of combinatorial games; these are games in which one player has options if and only if the other player does. They are commonly referred to as *all-small games* in normal play, since they are infinitesimal with respect to numbers.

All-small games are often challenging to study in detail because of their complex canonical forms. Since the 1970s, the main tool for analysing them has been approximation by *atomic weight* (Berlekamp et al., 2001; Conway, 2001); one can define an equivalence relation whose classes are assigned descriptive values that, in many cases, are sufficient to determine game outcomes and to provide a useful method for finding good moves. Later, in the 1980s, although never formally published, John Horton Conway and Alex Ryba proposed the theory of *uptimals*, which is finer and more precise than atomic weight theory (Albert et al., 2019, p. 210). The powers of $\uparrow$ arise frequently in game practice, and uptimals are precisely the games that can be expressed as sums of powers of $\uparrow$ and $*$. McKay (2011) provides a comprehensive analysis of this theory, establishing in particular that uptimals are totally ordered, characterising their canonical forms, developing methods for handling them in play, and giving numerous examples of all-small rulesets in which uptimals play a prominent role.

We show that, subject to certain conditions, these results admit a parallel under the misère play convention. The recent development of absolute combinatorial game theory makes this analysis possible: by considering different ranges of distinguishing games, one can perform effective comparison tests (Larsson et al., 2025b). Here, we are primarily interested in the universe of dicotic misère games $\mathcal{D}$, and secondarily the full misère universe $\mathcal{M}$, which contains all possible game forms.

In Section 2, we show that $\mathcal{D}$ contains a substructure similar to that of the uptimals. Alongside the main results, we give examples of game practice using the ALL-SMALL SHOVE ruleset. In Section 3, we show that there exist *tame dicotic rulesets* in which the atomic weight theory developed for normal

play can also be applied in *misère*. In other words, just like in tame impartial rulesets, players can play under the *misère* play convention using the two-ahead rule as if it were normal play, until reaching certain well-described endgames determined in advance. Larsson and Nowakowski (2023) cited personal correspondence with one of the present authors when they briefly presented the notion of atomic weight tameness. At the time, no published account was available to be cited; this paper supplies that account. A finer analysis of related positions, resting on the evil twin property, was carried out by Rubinstein-Salzedo and Zhou (2026), who ask whether genus theory can be generalised to partizan *misère* games. Their outcome correspondence is much stronger than anything we obtain; our framework is deliberately coarser, asking only that a normal play winning strategy survive the change of convention. Three case studies are presented, namely ALL-SMALL SHOVE, FLOWER GARDENS, and POISONED HACKENBUSH, the last of which is not tame, but contains a large tame subclass. In Section 4, we show that these results lose their effectiveness in $\mathcal{M}$, the full universe of all possible game forms, which is not Archimedean and contains no infinitesimals.

We assume that the reader is acquainted with the fundamental concepts of combinatorial game theory as presented in any of (Albert et al., 2019; Berlekamp et al., 2001; Siegel, 2013). We will freely use standard game forms, the normal play comparison test, and related tools; since the main focus is *misère* theory, we will write $\geq^+$ (and similar terms) to denote the normal play relation, reserving $\geq$ for the *misère* equivalent. The players are Left and Right, the ‘well-known’ pair in CGT.¹ Nevertheless, for the sake of self-containment, we conclude this introduction with a brief subsection presenting the results that will be most frequently used throughout this article.

1.1. Some background

The notion of atomic weight rests on a relaxation of game equivalence. We write $G \sim_\star H$, and say that G and H are *remote star equivalent*, if $G + X + \star n$ and $H + X + \star n$ have the same outcome, for every game X and all sufficiently large n . It is convenient to introduce the symbol $\star$ (pronounced ‘remote star’) to mean ‘ $\star n$ for n sufficiently large’.

Some algebraic properties of $\star$ hold, such as $\star + \star = \star$ or $\star n + \star = \star$

¹Left uses she/her pronouns and is the positive player, while Right uses he/him pronouns and is the negative player.

for any nonnegative integer n . Moreover, even though we have $\uparrow + * \in \mathcal{N}$, $\uparrow + *n \in \mathcal{L}$ holds for all $n \neq 1$; for this reason, we write $\uparrow \star \in \mathcal{L}$.²

Clearly, $G \sim_\star H$ may hold with $G \not\equiv^+ H$. For example, $\uparrow = \{0 \mid *\}$ is not equivalent to $H = \{0, *, *2 \mid 0, *\}$, yet $\uparrow \sim_\star H$. In fact, both games have atomic weight one, which means that, in disjunctive sums containing a remote star, Left has an advantage in each of these games corresponding to one waiting move.

The following theorem yields a constructive test for $G \sim_\star H$ and will be used here in several proofs. For this theorem and the ones that follow in this subsection, we provide references for the proofs.

Theorem 1.1 (cf. (Albert et al., 2019, p. 217)). *If G and H are all-small games, then $G \sim_\star H$ if and only if*

$$\downarrow \star <^+ G - H <^+ \uparrow \star.$$

In particular, $\text{aw}(G) = n$ if and only if $\downarrow \star <^+ G - n \cdot \uparrow <^+ \uparrow \star$.

In practical play, players should aim to gain an advantage in atomic weight so that they can exploit the two-ahead rule, which we now state. Crucially, atomic weight is additive with respect to the disjunctive sum: $\text{aw}(G + H) = \text{aw}(G) + \text{aw}(H)$, which greatly facilitates the application of the two-ahead rule.

Theorem 1.2 (Two-Ahead Rule, cf. (Albert et al., 2019, p. 221)).

If $\text{aw}(G) \geq 2$, then $G >^+ 0$.

If $\text{aw}(G) \geq 1$, then $G \triangleright^+ 0$.

If $\text{aw}(G) \leq -1$, then $G \triangleleft^+ 0$.

If $\text{aw}(G) \leq -2$, then $G <^+ 0$.

For the misère play convention, we will likewise use a constructive comparison test, described in the following theorem.

Theorem 1.3 (cf. (Larsson et al., 2025a, p. 250), (Larsson et al., 2021)). *If $\mathcal{D}$ is the universe of dicotic misère games and $G, H \in \mathcal{D}$, then $G \geq_{\mathcal{D}} H$ if and only if the following conditions are satisfied:*

²Recall that the *multiplicative notation* GH is sometimes used to denote the sum $G + H$. Accordingly, in this context, $\uparrow \star$ stands for $\uparrow + \star$.

Proviso: If $H = 0$, then Left, playing first, must win G ; if $G = 0$, then Right, playing first, must win H ;

Maintenance: For each Right option G^R , either there exists a Right option H^R such that $G^R \geq_{\mathcal{D}} H^R$, or there exists a Left option G^{RL} such that $G^{RL} \geq_{\mathcal{D}} H$;

For each Left option H^L , either there exists a Left option G^L such that $G^L \geq_{\mathcal{D}} H^L$, or there exists a Right option H^{LR} such that $G \geq_{\mathcal{D}} H^{LR}$.

A consequence of this comparison test is that $\{ * \mid * \} \equiv_{\mathcal{D}} 0$. Since $\{ * \mid * \} = * + *$, it follows that $* + * \equiv_{\mathcal{D}} 0$. Moreover, $G \geq_{\mathcal{D}} H$ implies that $G \geq^+ H$ under the normal play convention. This is justified by the fact that Left can use the maintenance property to win $G - H$ when playing second.

As shown in (Dorbec et al., 2015, Lemma 3.2), an essential end-reversible option—essential in the sense of being the only winning move when the game is played in isolation—can be replaced by $*$. We recall that an end-reversible option G^L is such that there exists $G^{LR} = \{ \cdot \mid G^{LRR} \} \leq_{\mathcal{D}} G$. Apart from this result, canonical forms in $\mathcal{D}$ are computed exactly as in normal play. Finally, Larsson et al. (2025a, p. 262) establishes that when a game G is invertible modulo $\mathcal{D}$, its inverse is its conjugate $\overline{G}$, i.e. the game form obtained by interchanging Left and Right. A necessary and sufficient criterion for a dicot to be invertible modulo $\mathcal{D}$ is given in (Fisher et al., 2022); more recently, Davies and Yadav (2025) have shown that this conjugate property holds in every universe.

In the full misère universe, the proviso becomes more restrictive, as stated in the next theorem. For example, it is no longer true that $* + *$ is equal to 0 modulo $\mathcal{M}$. Nevertheless, it remains true that $G \geq_{\mathcal{M}} H$ implies that $G \geq^+ H$ under the normal play convention. Indeed, this is an absolute result, since the maintenance property is included in the constructive comparison test of every universe.

Theorem 1.4 (cf. (Siegel, 2013, p. 275)). *If $\mathcal{M}$ is the full misère universe and $G, H \in \mathcal{M}$, then $G \geq_{\mathcal{M}} H$ if and only if the following conditions hold:*

Proviso: If H is a Left end, then so is G ; if G is a Right end, so is H ;

Maintenance: Formulated in the same way as in Theorem 1.3, but with $\geq_{\mathcal{M}}$.

2. Mups, mowns, powers of mups, and powers of mowns

We begin by describing the rules of ALL-SMALL SHOVE, which is a dicotic ruleset introduced in (Albert et al., 2019); we will use it to illustrate several concepts in this section.

A component is a finite strip of squares, each being either empty or occupied by a **bLack** (**L**eft) or **gRey** (**R**ight) piece. On their turn, a player selects one of their pieces and pushes it, together with all pieces to its left, one square to the left. Pieces may be pushed off the end of the strip. The play ends in a component when only pieces of one colour remain. When a player has no moves available in any component on their turn, the winner is then determined according to either the normal play or misère play convention. Figure 1 shows an ALL-SMALL SHOVE strip, together with the options available to each player.

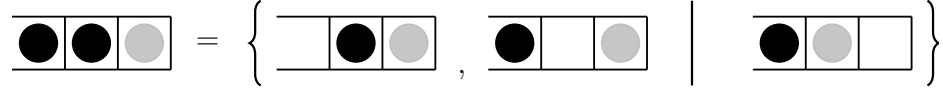


Figure 1: Up or mup, depending on the winning convention.

This first example is not arbitrary, as its literal form underpins all subsequent analysis. As such, we give a definition and terminology for it.

Definition 2.1 (cf. (Larsson et al., 2018)). The game form $\uparrow = \{0, * \mid *\}$ is referred to as *mup* (‘misère up’), while $\downarrow = \{*\mid 0, *\}$ is referred to as *mown* (‘misère down’). The game forms $\{0, * \mid 0\}$ and $\{0 \mid 0, *\}$ are called *mup star* and *mown star*, respectively.

The names, and the symbols $\uparrow$ and $\downarrow$, are those introduced in (Larsson et al., 2018), where the partial order of the dicotic misère games born by day 2 was also determined; we follow that notation throughout.

The position shown in Figure 1 is identified with the game form $\uparrow$. Left has the options to zero or star, whereas Right can only move to star. Under the normal play convention, Left’s option to star reverses out, so the form simplifies to $\uparrow = \{0 \mid *\}$. As for the misère play convention, the situation is quite different. Since it is the only winning move when the game is played in isolation, the option to star is end-reversible, yet it cannot be removed from the game form. In other words, in normal play, $\uparrow$ and $\uparrow$ belong to the same

equivalence class. However, the equivalence class splits in misère play, and the two games are no longer equivalent. Left should prefer $\uparrow$, since there are situations in which it makes the difference between winning and losing.³

Figure 2, adapted from (Larsson et al., 2018), shows the partial order of the dicotic misère games born by day 2; $\uparrow$ and $\uparrow + *$ are the only two with no games above them. Mup is the simplest positive game. Observe also that, under the misère play convention, outcome classes do not match the partial order: $\{0, * \mid 0\} \not\geq_{\mathcal{D}} 0$, but $\{0, * \mid 0\} \in \mathcal{L}$; on the other hand, $\{0, * \mid *\} \geq_{\mathcal{D}} 0$, but $\{0, * \mid *\} \in \mathcal{N}$.

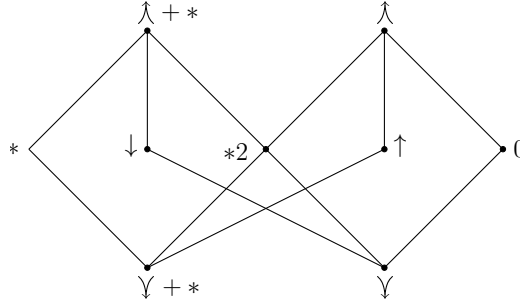


Figure 2: Partial order of the dicotic misère games born by day 2, adapted from (Larsson et al., 2018).

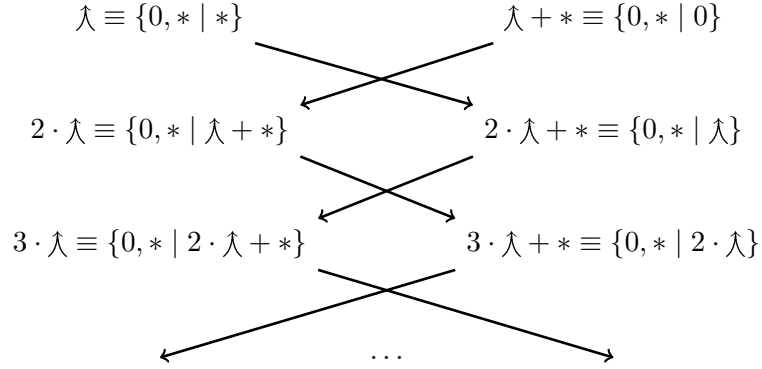
We will see that, when a ruleset admits positions of the literal form $\uparrow = \{0 \mid *\}$, the analyses tend to be quite intricate. By contrast, rulesets that do not admit this form, containing only positions identified with $\uparrow$, can sometimes lead to interesting analyses in misère. For instance, if the position in Figure 1 were a CLOBBER position (Albert et al., 2019), its game form would be $\uparrow$. Indeed, CLOBBER is notoriously difficult to analyse under the misère play convention.

In this section, for ease, we write $\geq$ and $\equiv$ instead of $\geq_{\mathcal{D}}$ and $\equiv_{\mathcal{D}}$, beginning with some preliminary results. The reader should keep in mind that everything takes place within the universe $\mathcal{D}$.

Theorem 2.2 (Copies of mups and mowns). *If n is a nonnegative integer, then the following facts hold:*

³The choice of the symbol ' $\uparrow$ ' was made with the refinement of the equivalence class of $\uparrow$ in mind. Indeed, the base of the symbol branches, just as the equivalence class splits.

- (i) $\{\lambda, 0, \forall\} \subseteq \mathcal{N}$ and $\ast \in \mathcal{P}$;
- (ii) $n \geq 1 \implies n \cdot \lambda + \ast \in \mathcal{L}$ and $n \cdot \forall + \ast \in \mathcal{R}$;
- (iii) $n > 1 \implies n \cdot \lambda \in \mathcal{L}$ and $n \cdot \forall \in \mathcal{R}$;
- (iv) $n > 0 \implies n \cdot \lambda > 0$ and $n \cdot \forall < 0$;
- (v) $n \cdot \lambda$ and $n \cdot \forall$ are invertible;
- (vi) The canonical forms of $n \cdot \lambda$ and $n \cdot \lambda + \ast$ satisfy the following recursion:



Proof. For item (i), in both mup and mown, each player has a winning move to star. Zero is trivially an $\mathcal{N}$ -position, since there are no moves. Star is a $\mathcal{P}$ -position because the players are forced to play the losing move to zero.

For item (ii), let $n = 1$, so that the position is $\{0, \ast \mid \ast\} + \ast$. Playing first, Left wins by moving to $\ast$. When playing second, against Right's moves to $\ast + \ast$ or to λ , Left can win in both cases by responding with $\ast$.

If $n > 1$, then, by induction, Left wins by moving to $(n - 1) \cdot \lambda + \ast$. If Left plays second, Right may move to $n \cdot \lambda$ or to $(n - 1) \cdot \lambda + \ast + \ast$. In both cases, Left wins by making exactly the same move to $(n - 1) \cdot \lambda + \ast$.

For item (iii), if Left moves first, she has a move to $(n - 1) \cdot \lambda + \ast$ with $n \geq 1$. If Right moves first, his only available move is also to $(n - 1) \cdot \lambda + \ast$. In both cases, as shown in item (ii), Left wins.

For item (iv), if $n = 1$, the game under consideration is λ . In this case, it is straightforward to see that the proviso is satisfied, since Left wins when playing first by moving to star. Moreover, maintenance holds, as against Right's only move to star, Left can revert to zero. The inequality is strict,

since λ is not equal to zero, given that Left's move to zero cannot be reversed. If $n > 1$, the rest follows by induction. By item (ii), the proviso is satisfied, as Left wins when playing first by moving to $(n-1) \cdot \lambda + *$. The maintenance holds, as against Right's only move to $(n-1) \cdot \lambda + *$, Left can revert to $(n-1) \cdot \lambda$. Naturally, the inequality is strict since the game is non-zero, being an $\mathcal{L}$ -position. It follows that $n \cdot \lambda > 0$, and the argument for $n \cdot \gamma < 0$ is symmetric.

For item (v), if $n = 0$, the result is trivial since 0 is its own inverse. If $n = 1$, then we have to check whether $\lambda + \gamma \equiv 0$. It is clear that the proviso is verified since, when playing first, Left wins by moving to $\lambda + *$, and Right also wins by moving to $\gamma + *$. On the other hand, against any move by either player, the opponent can revert to zero, whether in the terminal form $\{\cdot \mid \cdot\}$ or in the form $* + *$. Consequently, the maintenance holds from the perspective of either player, and $\lambda + \gamma \equiv 0$. For $n > 1$, it follows that $n \cdot \lambda + n \cdot \gamma \equiv n \cdot (\lambda + \gamma) \equiv n \cdot 0 = 0$.

For item (vi), we prove only that the canonical form of $n \cdot \lambda$ is $\{0, * \mid (n-1) \cdot \lambda + *\}$, since the proof for $n \cdot \lambda + *$ is analogous.

For $n = 1$, the equality is precisely the one from the definition. Thus, let $n > 1$ and note that the literal form of $n \cdot \lambda$ is

$$\{(n-1) \cdot \lambda, (n-1) \cdot \lambda + * \mid (n-1) \cdot \lambda + *\}.$$

Let us check that both choices of Left in this form are reversible. As for the first one, it suffices to verify the inequality

$$n \cdot \lambda \geq (n-2) \cdot \lambda + *.$$

Using item (v), terms cancel, and it suffices to verify the inequality

$$2 \cdot \lambda \geq *.$$

The maintenance is satisfied because

- (i) Left can reverse a Right move on the left side to $\lambda + *$ in the same component, yielding the inequality $* \geq *$;
- (ii) The reversion of a Left move on the right side to 0 can be made on the left side, yielding, by item (iv), the inequality $\lambda \geq 0$.

As for the second one, it suffices to verify the inequality

$$n \cdot \lambda \geq (n-1) \cdot \lambda.$$

This time, item (v) and cancellation lead to $\lambda \geq 0$, and, taking item (iv) into account, there is nothing left to verify.

Thus, the options are reversed via $(n-2) \cdot \lambda + *$ and $(n-1) \cdot \lambda$. If $n = 2$, the former is $0 \cdot \lambda + * = *$, whose only Left option is 0; if $n > 2$, the two games are equivalent, by induction, to $\{0, * \mid (n-3) \cdot \lambda\}$ and $\{0, * \mid (n-2) \cdot \lambda + *\}$, respectively. In either case, the Left options supplied by the reversals are 0 and $*$, and we obtain that $n \cdot \lambda \equiv \{0, * \mid (n-1) \cdot \lambda + *\}$. In this form, $*$ is an end-reversible option for Left and cannot be removed. The Right option is not reversible either: its Left options are 0 and $*$, and neither is greater than or equal to $n \cdot \lambda$. Indeed, $n \cdot \lambda > 0$ by item (iv); and $* \geq n \cdot \lambda$ would give $* \geq 2 \cdot \lambda$, which, together with $2 \cdot \lambda \geq *$, would force $* \equiv 2 \cdot \lambda$. This is impossible since $* \in \mathcal{P}$ while $2 \cdot \lambda \in \mathcal{L}$. Therefore, this is a canonical form and the proof is complete. $\square$

Mups and mowns form the basis of a recursive ‘net’ of infinitesimals. Conceptually, this substructure works because Left has a choice between 0 and $*$ in endgames. After the following definition, we can establish some relevant results.

Definition 2.3. We define the *powers of mup* and the *powers of mown* recursively as follows:

$$\begin{aligned} \lambda^1 &:= \lambda = \{0, * \mid *\}, \\ \lambda^n &:= \{0, * \mid \forall + \forall^2 + \cdots + \forall^{n-1} + *\}, \quad \text{for } n > 1, \\ \forall^1 &:= \forall = \{*\mid 0, *\}, \\ \forall^n &:= \{\lambda + \lambda^2 + \cdots + \lambda^{n-1} + * \mid 0, *\}, \quad \text{for } n > 1. \end{aligned}$$

Theorem 2.4. *If $n \geq 1$, then we have the following facts:*

- (i) $\lambda^n + * \in \mathcal{L}$, $\forall^n + * \in \mathcal{R}$, and $\{\lambda^n, \forall^n\} \subseteq \mathcal{N}$;
- (ii) $\lambda^n > 0$ and $\forall^n < 0$;
- (iii) λ^n and $\forall^n$ are invertible.

Proof. The proof proceeds by induction on n over all items simultaneously.

Starting with item (i) and the particular case $\lambda^n + * \in \mathcal{L}$, if $n = 1$, then the result follows directly from the definition. Let us assume that $n > 1$. Playing first, Left wins by moving to $*$. If Right plays first and moves to λ^n , then Left wins by replying with $*$. If Right moves to

$$\vee + \vee^2 + \cdots + \vee^{n-1} + * + *,$$

then, by induction and invertibility, Left wins by responding with

$$\vee + \vee^2 + \cdots + \vee^{n-2} + * + \wedge + \wedge^2 + \cdots + \wedge^{n-2} + * + * \equiv *.$$

The argument for $\vee^n + * \in \mathcal{R}$ is analogous. It is also straightforward to verify that λ^n is an $\mathcal{N}$ -position. It suffices to consider $n > 1$, since the case $n = 1$ follows directly from the definition. If Left moves first, she wins by moving to $*$. If Right moves first, he wins by moving to $\vee + \vee^2 + \cdots + \vee^{n-1} + *$. This is a winning move, since by induction we have $\vee^{n-1} + * \in \mathcal{R}$ and $\vee + \vee^2 + \cdots + \vee^{n-2} < 0$. The argument for $\vee^n \in \mathcal{N}$ is analogous.

For item (ii), if $n = 1$, the result follows from Theorem 2.2. Hence, let us consider $n > 1$ and the game

$$\{0, * \mid \vee + \vee^2 + \cdots + \vee^{n-1} + *\}.$$

Left's winning option is $*$. Against Right's only option, Left replies with

$$\wedge + \wedge^2 + \cdots + \wedge^{n-2} + * + \vee + \vee^2 + \cdots + \vee^{n-2} + *.$$

By induction, this sum is equivalent to zero, and the maintenance is verified. Thus, we have $\lambda^n \geq 0$. Finally, the inequality must be strict, since from Right's point of view, Left's move to 0 cannot be reversed. As desired, we have shown that $\lambda^n > 0$, and the argument for $\vee^n < 0$ is symmetric.

Finally, for item (iii), consider the game

$$\lambda^n + \vee^n = \{0, * \mid \vee + \vee^2 + \cdots + \vee^{n-1} + *\} + \{\wedge + \wedge^2 + \cdots + \wedge^{n-1} + * \mid 0, *\}.$$

That this sum is equivalent to zero when $n = 1$ is a direct consequence of Theorem 2.2. We therefore assume that $n > 1$. If Left moves first, she wins by moving to $\wedge + \wedge^2 + \cdots + \wedge^{n-1} + \lambda^n + *$. This is a winning move, since by induction we have $\lambda^n + * \in \mathcal{L}$ and $\wedge + \wedge^2 + \cdots + \wedge^{n-1} > 0$. If Right moves

first, he has a winning move that is symmetric to this one. Against Right's moves to

$$\lambda^n + *, \lambda^n + 0, \text{ or } \forall + \forall^2 + \cdots + \forall^{n-1} + * + \forall^n,$$

Left may reverse to

$$* + *, 0 + 0, \text{ or } \forall + \forall^2 + \cdots + \forall^{n-1} + * + \lambda + \lambda^2 + \cdots + \lambda^{n-1} + *,$$

all equivalent to zero. Thus, the maintenance holds from Left's perspective, and the scenario is symmetric for Right. We conclude that $\lambda^n + \forall^n \equiv 0$, which completes the proof. $\square$

Definition 2.5 (cf. (Larsson et al., 2016, Observation 17)). Let Λ be a partially ordered monoid and let $a, b \in \Lambda$ with $0 < a < b$. We say that a is *infinitesimal with respect to b* if $k \cdot a < b$ for every positive integer k , and that Λ *contains infinitimals* if such a pair exists.

Theorem 2.6. *If $n \geq 2$, then λ^n is infinitesimal with respect to λ^{n-1} .*

Proof. Fix $n \geq 2$ and put

$$A_k = k \cdot \forall^n + \lambda^{n-1} \quad \text{and} \quad B_k = A_k + \lambda + \lambda^2 + \cdots + \lambda^{n-1} + *.$$

We prove $A_k \geq 0$ and $B_k \geq 0$ simultaneously by induction on k . The first inequality for $k = 0$ is Theorem 2.4. For $k \geq 1$, Left establishes the proviso for A_k by moving in a $\forall^n$ component to B_{k-1} , since

$$B_{k-1} = \underbrace{(k-1) \cdot \forall^n + \lambda^{n-1}}_{\geq 0 \text{ (induction)}} + \underbrace{\lambda^2 + \cdots + \lambda^{n-1}}_{\geq 0 \text{ (Theorem 2.4)}} + \underbrace{\lambda + *}_{\in \mathcal{L} \text{ (Theorem 2.2)}} \geq \lambda + *.$$

For maintenance, Right's three types of move lead to A_{k-1} , $A_{k-1} + *$, or $k \cdot \forall^n + \forall + \cdots + \forall^{n-2} + *$. Left responds, respectively, by moving to 0 if $k = 1$ and to B_{k-2} otherwise, by removing the star, or by moving in a $\forall^n$ component. Cancellation leaves a nonnegative position in each case, so $A_k \geq 0$.

For B_k , Left establishes the proviso by moving in the λ component to 0; the result is $X + * \geq *$, where $X = A_k + \lambda^2 + \cdots + \lambda^{n-1} \geq 0$. For maintenance, a Right move in a $\forall^n$ component is paired with the appropriate move in λ ; a move in λ^j , for $j < n-1$, with a move in λ^{j+1} ; and a move in the external star with the move from λ to 0. The remaining case, a move in either copy

of λ^{n-1} , leaves A_k after cancellation; Left moves to B_{k-1} if $k > 0$, and to 0 if $k = 0$. The induction therefore gives $B_k \geq 0$.

If $A_k \equiv 0$, then $A_{k+1} \equiv \downarrow^n < 0$, contradicting the weak inequality just proved. By invertibility, $k \cdot \lambda^n < \lambda^{n-1}$ for every $k > 0$, as required. $\square$

For the canonical forms of powers of mup and mown, we can also obtain a complete characterisation, as in normal play.

Theorem 2.7. *Let $\lambda^{[0]} = 0$, and for $n > 0$, let $\lambda^{[n]} = \{*, \lambda^{[n-1]} \mid *\}$. Moreover, for $n > 0$, we write $\lambda^{[n]}*$ for the form $\{0, \lambda^{[n-1]}* \mid 0\}$. The following facts hold:*

- (i) $\lambda + \lambda^2 + \cdots + \lambda^n \equiv \lambda^{[n]}$;
- (ii) $\lambda + \lambda^2 + \cdots + \lambda^n + * \equiv \lambda^{[n]}*$;
- (iii) $\lambda^{[n]}$ and $\lambda^{[n]}*$, defined recursively as in the statement of the theorem, are canonical forms;
- (iv) The canonical forms of the games λ^n and λ^n* are $\{0, * \mid \downarrow^{[n-1]}*\}$ and $\{0, * \mid \downarrow^{[n-1]}\}$, respectively.

These facts remain valid under symmetry with respect to Right.

Proof. Items (i) and (ii) are proved by simultaneous induction on n ; the case $n = 1$ is Definition 2.1, so we assume that $n > 1$.

In $\sum_{i=1}^n \lambda^i$, the moves in λ^n dominate all corresponding moves in lower powers, by Theorem 2.6. Thus the sum is equivalent to

$$G = \left\{ * + \sum_{i=1}^{n-1} \lambda^i, \sum_{i=1}^{n-1} \lambda^i \mid * \right\}.$$

By induction, $G^{L_1} \equiv \lambda^{[n-1]}* \equiv \{0, \lambda^{[n-2]}* \mid 0\}$. Also $G \geq 0$: G^{L_1} wins for Left, while Right's move to $*$ can be answered by a move to 0. This is Left's unique winning option. Indeed, $G^{L_1} \geq \lambda* \in \mathcal{L}$, whereas after $G^{L_2} = \sum_{i=1}^{n-1} \lambda^i$, Right moves in λ^{n-1} and leaves $*$. Since the Right option of G^{L_1} is $0 \leq G$, it is an essential end-reversible option and is replaced by $*$. Induction now gives

$$G \equiv \{*, \lambda^{[n-1]} \mid *\}.$$

Similarly, removing dominated options from $* + \sum_{i=1}^n \lambda^i$ leaves

$$H = \left\{ * + \sum_{i=1}^{n-1} \lambda^i, \sum_{i=1}^n \lambda^i \mid 0 \right\}.$$

By item (i), $\sum_{i=1}^n \lambda^i \equiv \{*, \lambda^{[n-1]} \mid *\}$, and the comparison test gives $H \geqslant *$. Hence the second Left option is reversible and is replaced by 0; the first is $\lambda^{[n-1]}*$ by induction. Therefore

$$H \equiv \{0, \lambda^{[n-1]}* \mid 0\}.$$

For item (iii), $*$ and $\lambda^{[n-1]}$ are incomparable in normal play, so also modulo $\mathcal{D}$. Moreover, $G \not\leqslant^+ 0$ and $G \not\leqslant^+ *$, whence $G \not\leqslant_{\mathcal{D}} 0$ and $G \not\leqslant_{\mathcal{D}} *$. There is therefore no domination or reversibility in $\{*, \lambda^{[n-1]} \mid *\}$ beyond Left's essential end-reversible option $*$. Since $\lambda^{[n-1]}$ is in canonical form by induction, so is $\{*, \lambda^{[n-1]} \mid *\}$; the argument for $\{0, \lambda^{[n-1]}* \mid 0\}$ is the same.

Item (iv) follows by ruling out domination and reversibility in the same way in $\{0, * \mid \forall^{[n-1]}*\}$ and $\{0, * \mid \forall^{[n-1]}\}$. $\square$

With the net of infinitesimals established by the powers of mups and mowns, we can rigorously describe a wide range of game values, just as in normal play. After defining this structure, we will see that there is a set of ALL-SMALL SHOVE strips with game values of this type.

Definition 2.8. Let $d_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant n$ and

$$.d_1 d_2 \dots d_n := d_1 \cdot \lambda + d_2 \cdot \lambda^2 + \dots + d_n \cdot \lambda^n.$$

We say that $G = .d_1 d_2 \dots d_n$ is a *muptimal* written in *muptimal notation*. We call the d_i 's *coefficients*, and negative coefficients are indicated by a bar.

Observation 2.9. When G is written in muptimal notation, the representation of $\overline{G}$ is obtained by removing the bar from the digits that have it and placing a bar over the digits that do not.

For instance, if $G = .2\overline{1}43$, then G is the sum $2 \cdot \lambda + \forall^2 + 4 \cdot \forall^3 + 3 \cdot \lambda^4$ and $\overline{G} = .\overline{2}14\overline{3}$ is the sum $2 \cdot \forall + \lambda^2 + 4 \cdot \lambda^3 + 3 \cdot \forall^4$.

Observation 2.10. Let $G = .d_1d_2\dots d_k\dots d_n$ be a muptimal written in muptimal notation. If $d_k > 0$, Left can move to

$$G^{L_1} = .d_1d_2\dots(d_k - 1)d_{k+1}\dots d_n \text{ or}$$

$$G^{L_2} = .d_1d_2\dots(d_k - 1)d_{k+1}\dots d_n + *.$$

If $d_k < 0$, Left can move to

$$G^{L_3} = .(d_1 + 1)(d_2 + 1)\dots(d_k + 1)d_{k+1}\dots d_n + *.$$

For instance, in $G = .2\bar{1}43$, regarding the positive digits, Left has the options $G^{L_1} = .1\bar{1}43$, $G^{L_2} = .1\bar{1}43 + *$, $G^{L_3} = .2\bar{1}42$, and $G^{L_4} = .2\bar{1}42 + *$. Considering the negative digits, Left has $G^{L_5} = .30\bar{4}3 + *$ and $G^{L_6} = .30\bar{3}3 + *$. Some of these options are dominated.

Theorem 2.11. *If $n \geq 1$ and $k \geq 1$, then*

$$\uparrow^n > d_1 \cdot \uparrow^{n+1} + \dots + d_k \cdot \uparrow^{n+k},$$

where $d_i \in \mathbb{Z}$ for $1 \leq i \leq k$.

Proof. The proof proceeds by induction on k . If $k = 1$ and $d_1 < 0$, we have

$$\uparrow^n > 0 > d_1 \cdot \uparrow^{n+1}.$$

If $k = 1$ and $d_1 \geq 0$, then the result is a consequence of Theorem 2.6. Hence, we may assume that $k > 1$.

On one hand, by induction, we have

$$\uparrow^{n+1} > d_2 \cdot \uparrow^{(n+1)+1} + \dots + d_k \cdot \uparrow^{(n+1)+k-1} = d_2 \cdot \uparrow^{n+2} + \dots + d_k \cdot \uparrow^{n+k}.$$

On the other hand, we have just seen that

$$\uparrow^n > (d_1 + 1) \cdot \uparrow^{n+1} = d_1 \cdot \uparrow^{n+1} + \uparrow^{n+1}.$$

Together, these two facts yield the inequality

$$\uparrow^n > (d_1 + 1) \cdot \uparrow^{n+1} = d_1 \cdot \uparrow^{n+1} + \uparrow^{n+1} > d_1 \cdot \uparrow^{n+1} + d_2 \cdot \uparrow^{n+2} + \dots + d_k \cdot \uparrow^{n+k}.$$

□

Corollary 2.12. *Muptimals are totally ordered.*

Proof. Given two different muptimals $G = .g_1g_2 \dots g_n$ and $H = .h_1h_2 \dots h_m$, they must differ in at least one coefficient. Consider the leftmost such coefficient, and assume that it is the coefficient of λ^n , i.e. the first difference from left to right occurs at position n . Without loss of generality, assume that G has the larger coefficient at that position, and we will show that $G > H$. It follows from Theorem 2.4 that muptimals are invertible, so it suffices to prove that $G + \overline{H} > 0$. Hence, the inequality to be verified is $.0 \dots 0(g_n - h_n) \dots > 0$, that is, $.0 \dots 0(g_n - h_n) > .0 \dots 00(h_{n+1} - g_{n+1}) \dots$. As $g_n - h_n \geq 1$, we have

$$\begin{aligned} .0 \dots 0(g_n - h_n) &\geq .0 \dots 01 && (\text{Theorem 2.4}) \\ &= \lambda^n \\ &> (h_{n+1} - g_{n+1}) \cdot \lambda^{n+1} + \dots && (\text{Theorem 2.11}) \\ &= .0 \dots 00(h_{n+1} - g_{n+1}) \dots \end{aligned}$$

which completes the proof. $\square$

Theorem 2.13. *Let $G = .d_1d_2 \dots d_n$ be a muptimal written in muptimal notation, where the leftmost non-zero digit is positive. Then, we have the following facts:*

- (i) *if the initial digits consist of a (possibly empty) string of 1's followed by a digit greater than or equal to 2—that is, if $d_1 = \dots = d_m = 1$ and $d_{m+1} \geq 2$ for some $m \geq 0$ —then $G \in \mathcal{L}$; otherwise, we have $G \in \mathcal{N}$;*
- (ii) $G + * \in \mathcal{L}$.

A symmetric result holds when the leftmost non-zero digit is negative.

Proof. We prove items (i) and (ii), together with their conjugates, by simultaneous induction on the formal birthday.

For item (i), suppose that Left starts. She moves in a mown component if there is one, and otherwise moves in a mup component to $*$. She reaches $*$, a $\mathcal{P}$ -position, or a position of item (ii) with positive leading digit; the latter is an $\mathcal{L}$ -position by induction, so in both cases her move wins.

Now suppose Right starts. If the initial digits have the form specified in item (i), a move in a mup component reduces an initial segment and introduces a star, while a move in a mown component leaves the initial positive pattern unchanged. In either case the first surviving digit is positive,

and induction gives Left a response. Otherwise, Right moves in a power of mup corresponding to the rightmost positive digit, d_k say, reaching

$$.(d_1 - 1) \dots (d_k - 1)d_{k+1} \dots d_n + *.$$

By the choice of k , the new digits are either all zero or have a negative leftmost non-zero entry. The conjugate of item (ii), or the fact that $* \in \mathcal{P}$, shows that this move wins for Right. This proves item (i).

For item (ii), if Left starts and a negative digit occurs, she moves in a corresponding mown component. The two stars cancel and the resulting muptimal begins with 1's followed by a digit at least 2. If there is no negative digit, she instead moves in a mup component to 0, leaving $*$ or a smaller position of item (ii). Either move wins by induction.

If Right starts, removing the external star leaves G , and a move in a mown component leaves the leading positive digit unchanged; induction handles both cases. A move in $\wedge^k$ cancels the external star and leaves

$$H = .(d_1 - 1) \dots (d_k - 1)d_{k+1} \dots d_n.$$

If H has no negative digit, induction applies (with the terminal case immediate). Otherwise, Left moves in its rightmost mown component. This restores the star and, by the choice of that component, restores a positive leftmost non-zero digit. The resulting position is covered by item (ii), completing the induction. $\square$

To illustrate an occurrence of muptimals, we return to ALL-SMALL SHOVE. Observe that, if the rightmost piece of a strip is grey, then the game ends as soon as the last black piece leaves the board. Thus, having one or more grey pieces to its right makes no difference to the play. Furthermore, the distance between this last black piece and the grey piece immediately to its right is irrelevant. Since the same observations apply symmetrically when the rightmost piece is black, without loss of generality, we restrict our attention to strips in which the last two pieces are adjacent and of opposite colours (Figure 3).

For ease of notation, ALL-SMALL SHOVE strips are represented by lists of positions. Positions are encoded by non-zero integers: positive integers denote squares occupied by black pieces, whereas negative integers denote those occupied by grey pieces. The latter are written using an overbar. For example, the strip in Figure 3, after merging and joining on the right, corresponds to the list $[1, \bar{3}, 4, \bar{5}]$.



Figure 3: Merging and joining on the right.

We observe that, if an ALL-SMALL SHOVE strip has its rightmost piece of one colour and all the others of the other colour, then its game value is a muptimal. To understand why, let us consider a muptimal such as .532. The only non-dominated Right option is .421 + *, and the only non-dominated Left options are .531 and .531 + *. In other words, we have

$$.532 \equiv \{.531, .531 + * \mid .421 + *\}.$$

It turns out that when Left plays to .531 + *, in a worst-case scenario, Right may respond with .531 and Left only worsens her position. This option is reversible and gives rise to a sequence of reductions allowing it to be replaced by *. In fact, we have

$$\{.531, g + * \mid .421 + *\} \equiv .532,$$

where g can be any nonnegative muptimal less than or equal to .531. In particular, which is relevant for the rest of this explanation, we have

$$\{.531, .421 + * \mid .421 + *\} \equiv .532.$$

Consider now a game value such as .632 + *. The only non-dominated Right option is .521, and the only non-dominated Left options are .631 + * and .632. In other words, we have

$$.632 + * \equiv \{.631 + *, .632 \mid .521\}.$$

It turns out that when Left plays to .632, in a worst-case scenario, Right may respond with .521 + * and Left only worsens her position. This option is reversible and gives rise to a sequence of reductions allowing it to be replaced by 0. In fact, we have

$$\{.631 + *, g \mid .521\} \equiv .632 + *,$$

where g can be any nonnegative muptimal less than or equal to .632. In particular, we have

$$\{.631 + *, .521 \mid .521\} \equiv .632 + *.$$

A consequence of these observations is that an ALL-SMALL SHOVE strip $[a_1, \dots, a_{k-2}, a_{k-1}, a_k]$, where a_k denotes the position of the unique grey piece (the only negative integer in the list), is equivalent to $.a_{k-2} \dots a_1 + a_{k-1} \cdot *$. The rightmost black piece is at position a_{k-1} ; if it is even, there should be no star in the sum; otherwise a star should appear. Only the parity of a_{k-1} matters, which is what the term $a_{k-1} \cdot *$ records.



Figure 4: $[2, 3, 5, 6, \bar{7}] \equiv .532$ and $[2, 3, 6, 7, \bar{8}] \equiv .632 + *$.

For example, consider the strip on the left in Figure 4, corresponding to $[2, 3, 5, 6, \bar{7}]$. This strip has value $\{.531, .421 + * \mid .421 + *\} \equiv .532$, where the sensible Left moves can be made either with the leftmost black piece or with the rightmost black piece. Right's option is unique.

The strip on the right in Figure 4, corresponding to $[2, 3, 6, 7, \bar{8}]$, has value $\{.631 + *, .521 \mid .521\} \equiv .632 + *$. Once again, the sensible Left moves can be made either with the leftmost black piece or with the rightmost black piece, and Right's option is unique.

3. Tame dicotic rulesets

Among ALL-SMALL SHOVE strips, those with a rightmost piece of one colour and all other pieces of the opposite colour are a rare case for which the misère game values can be computed efficiently. However, if we broaden our scope to include all other strips, the picture changes. Even with respect to the normal play convention, one can read in (Albert et al., 2019, p. 224): ‘(...) while some positions in this game are easily described using upitmal notation, others appear quite complex (...)’. Of course, the situation is no different under the misère play convention.

It turns out that in some rulesets, even when game values cannot be determined easily, one can still obtain useful information for practical play. In this section, we discuss cases where atomic weights arising from normal play can still be used in misère. To this end, it is useful to recall what happens in classical NIM, where it is well known that the misère play solution is essentially identical to the normal play solution, differing only by minor modifications:

Play just as you would in normal play NIM unless your move would leave a position consisting entirely of heaps of size 0 or 1. In that case, play to leave an odd number of heaps of size 1.

With this in mind, let us momentarily disregard the winning condition and focus on the following set of forms:

$$\mathcal{S} := \left\{ \sum_{i=1}^k *n_i \text{ such that there is at least one number of size larger than 1} \right\}.$$

This set of forms satisfies the *convention invariance property*: any form belonging to $\mathcal{S}$ has the same outcome under both normal play and misère play conventions. Yamasaki (1980) described a list of properties that a set of impartial forms,

$$\mathcal{F} := \mathcal{S} \sqcup \mathcal{F}^0 \sqcup \mathcal{F}^*,$$

should satisfy to use normal play strategies in misère play:

- (i) $\{\cdot \mid \cdot\} \in \mathcal{F}^0$;
- (ii) Every non-terminal form in $\mathcal{F}^0$ has at least one option in $\mathcal{F}^*$, and every form in $\mathcal{F}^*$ has at least one option in $\mathcal{F}^0$;
- (iii) No form in $\mathcal{F}^0$ has options in $\mathcal{F}^0$, and no form in $\mathcal{F}^*$ has options in $\mathcal{F}^*$;
- (iv) Every form in $\mathcal{S}$ has an option in $\mathcal{F}^0$ if and only if it has an option in $\mathcal{F}^*$;
- (v) Every option of a form belonging to $\mathcal{F}^0 \cup \mathcal{F}^*$ that goes into $\mathcal{S}$ has an option that returns to $\mathcal{F}^0 \cup \mathcal{F}^*$.

This list is better understood with the help of Yamasaki's diagram, as shown in Figure 5. Under the normal play convention, a player aims to move to $\mathcal{F}^0$; under the misère play convention, the aim is instead to move to $\mathcal{F}^*$. If no options in $\mathcal{F}^0 \cup \mathcal{F}^*$ are available, the play proceeds in the same way under both conventions. Thus, $\mathcal{S}$ satisfies the convention invariance property.

Conway (2001) and Berlekamp et al. (2001) indicated that there is a class of impartial rulesets, known as *tame*, that behave in this way. The *Genus Theory* was proposed, proving useful for many rulesets other than NIM, provided they obey the logic of Yamasaki's diagram. Roughly speaking, in a tame impartial ruleset, a player should play as in normal play until an opportunity arises to leave an odd number of components with genus 1^0 .

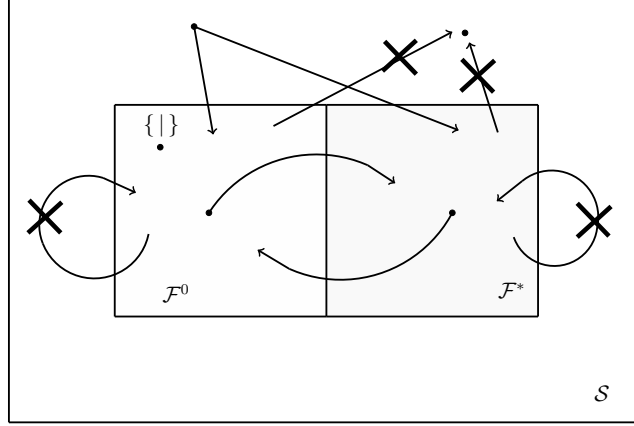


Figure 5: Yamasaki's diagram.

Regarding dicotic rulesets, we now show that a similar line of reasoning applies. There are rulesets in which Left (respectively, Right) can play as in normal play, maintaining an atomic weight greater than or equal to 2 (respectively, less than or equal to -2). When this is no longer possible, the player has a winning move in misère. In other words, just as players in tame impartial games deviate from the normal play strategy when they have the opportunity to leave an odd number of components of genus 1^0 , players in tame dicotic games deviate from the normal play strategy when they can no longer maintain a margin of two in terms of the normal play atomic weight. In that case, the tame nature of the ruleset guarantees the existence of a winning move in misère. This idea is captured by the following definition.

Definition 3.1. A *tame dicotic ruleset* is one in which every position P satisfies the following properties:

- (i) If $\text{aw}(P) \geq 2$ (under the normal play convention), any decrease of the atomic weight by Right is by at most one unit.
- (ii) If $\text{aw}(P) \geq 1$ and Left has no option P^L such that $\text{aw}(P^L) \geq 2$, she has a winning move in misère play.
- (iii) Items (i) and (ii) hold symmetrically for $\text{aw}(P) \leq -2$ and $\text{aw}(P) \leq -1$.

The definition gives the misère analogue of the normal play Two-Ahead Rule (Theorem 1.2).

Theorem 3.2 (Misère Two-Ahead Rule). *Let P be a position in a tame dicotic ruleset. Then*

- If $\text{aw}(P) \geq 2$, then $P \in \mathcal{L}$.*
- If $\text{aw}(P) \geq 1$, then $P \in \mathcal{L} \cup \mathcal{N}$.*
- If $\text{aw}(P) \leq -1$, then $P \in \mathcal{R} \cup \mathcal{N}$.*
- If $\text{aw}(P) \leq -2$, then $P \in \mathcal{R}$.*

Proof. We argue by induction on the birthday of P . If $\text{aw}(P) \geq 1$, then either Left has an option of weight at least two, which by induction is a winning move, or property (ii) of Definition 3.1 supplies one. Thus Left wins moving first. If $\text{aw}(P) \geq 2$, then, by property (i), every Right option has weight at least one, and so, by induction, is winning for Left. Hence $P \in \mathcal{L}$. The other two assertions are symmetric. $\square$

Observe that these properties do not hold for all game forms; a single position whose game value violates them makes a ruleset wild rather than tame. For example,

$$\text{aw}(\{\{5 \cdot \uparrow \uparrow * \}, \{5 \cdot \uparrow \uparrow \} \mid \{\uparrow * \downarrow * \}, \{\uparrow \downarrow * \}\}) = 2 \geq 2,$$

and Right can decrease the atomic weight to $\frac{1}{2}$. Thus, item (i) fails.

Also, $\text{aw}(\uparrow) = \text{aw}(\{0 \mid *\}) = 1 \geq 1$, and Left has no winning move under the misère play convention, so item (ii) fails. This observation reinforces what has already been said: the occurrence of the literal form of $\uparrow$ as a game value of a position in a ruleset is a symptom of its intractability. Likewise, $\text{aw}(\{0 \mid *, *2\}) = 1 \geq 1$, yet Left has no winning move under the misère play convention, so item (ii) fails in this case as well. Note that $n \cdot \{0 \mid *, *2\} \in \mathcal{R}$ for any n , so one can obtain positions of arbitrarily large atomic weight in normal play that remain $\mathcal{R}$ -positions in misère play.

Fortunately, there are many tame dicotic rulesets; indeed, Larsson and Nowakowski (2023, Theorem 20) showed, in effect, that BIPASS is tame. We present three further case studies below, beginning with ALL-SMALL SHOVE, this time considering all possible strips.

3.1. Case study 1: ALL-SMALL SHOVE

To investigate the tame nature of ALL-SMALL SHOVE, and since this issue is not addressed in (Albert et al., 2019), we first establish a general

result concerning the atomic weight of a strip. We recall that every ALL-SMALL SHOVE strip is assumed to have its two rightmost pieces adjacent and of opposite colours. Thus, for example, if a theorem concerns strips with exactly two pieces, then those two pieces are necessarily adjacent and of opposite colours.

In all results that follow, in a strip $[a_1, \dots, a_n]$, some a_i may be negative and overbars are not written explicitly. For convenience, whenever $a_i - 1$ is written to describe a leftward push, it should be understood that overbars are preserved. For example, if the position is $[a_1, a_2]$ with $a_1 = 2$ and $a_2 = \bar{5}$, then $[a_1 - 1, a_2 - 1]$ denotes a new strip with positions 1 and $\bar{4}$. In other words, the black piece in position 2 and the grey piece in position 5 are pushed to positions 1 and 4.

Theorem 3.3. *If $G = [a_1, a_2]$ is an ALL-SMALL SHOVE strip with two pieces, then $G \equiv^+ 0$ and $G \equiv_{\mathcal{D}} 0$ if a_1 is even, while $G \equiv^+ *$ and $G \equiv_{\mathcal{D}} *$ if a_1 is odd.*

Proof. Without loss of generality, suppose that $a_1 > 0$ and $a_2 < 0$ (the rightmost piece is grey). If $a_1 = 1$, then $G = \{[a_2] \mid [a_2 - 1]\}$; otherwise, $G = \{[a_1 - 1, a_2] \mid [a_1 - 1, a_2 - 1]\}$. In the first case, the options are terminal, so $G = \{0 \mid 0\}$; in the second, induction alternates between $\{* \mid *\}$ and $\{0 \mid 0\}$. These forms are equivalent to 0 and $*$, respectively, under both conventions. $\square$

Theorem 3.4. *If $G = [a_1, a_2, a_3]$ is an ALL-SMALL SHOVE strip with three pieces and $a_1 > 0$, then $G \equiv^+ a_1 \cdot \uparrow + a_2 \cdot *$, and $G \equiv_{\mathcal{D}} a_1 \cdot \blacktriangledown + a_2 \cdot *$. The symmetric result holds when $a_1 < 0$.*

Proof. We first work modulo $\mathcal{D}$ and argue by induction on a_1 . When $a_1 = 1$, Theorem 3.3 gives $G \equiv_{\mathcal{D}} \{0, * \mid *\}$ or $G \equiv_{\mathcal{D}} \{0, * \mid 0\}$, according as a_2 is even or odd. For $a_1 > 1$, induction gives

$$G \equiv_{\mathcal{D}} \begin{cases} \{(a_1 - 1) \cdot \blacktriangledown, (a_1 - 1) \cdot \blacktriangledown + * \mid (a_1 - 1) \cdot \blacktriangledown + *\}, & a_2 \text{ even,} \\ \{(a_1 - 1) \cdot \blacktriangledown, (a_1 - 1) \cdot \blacktriangledown + * \mid (a_1 - 1) \cdot \blacktriangledown\}, & a_2 \text{ odd.} \end{cases}$$

The reversals established in item (vi) of Theorem 2.2 reduce these to $a_1 \cdot \blacktriangledown$ and $a_1 \cdot \blacktriangledown + *$, respectively. Finally, equivalence modulo $\mathcal{D}$ implies normal play equivalence, and $\blacktriangledown \equiv^+ \uparrow$. $\square$

The next theorem concerns only the normal play convention.

Theorem 3.5. *If $G = [a_1, a_2, \dots, a_{k-2}, a_{k-1}, a_k]$ is an ALL-SMALL SHOVE strip with more than two pieces, then $\text{aw}(G) = a_{k-2}$.*

Proof. Without loss of generality, suppose that $a_{k-2} > 0$. Being a remote star, the last component may be taken to be $*n$ with $n \geq 2$.

The inequality

$$[a_1, \dots, a_{k-2}, a_{k-1}, a_k] + (a_{k-2} + 1) \cdot \downarrow + \star <^+ 0$$

holds because Right, regardless of whether playing first or second, has a winning strategy by always playing in the first component, as long as it contains at least three pieces. By playing his rightmost piece in that component, Right decreases a_{k-2} by one each time he plays. Left's replies elsewhere leave this count undisturbed: a move of hers in a down turns that down into a star, and a move in the remote star merely replaces one nimber by a smaller one. Therefore, there eventually comes a point when the first component has only two pieces left, and so, by Theorem 3.3, is 0 or $*$; at that moment at least one down survives, and the position is a sum of downs and nimbers. Right wins it, because

$$d \cdot \downarrow + *q <^+ 0 \quad \text{for } d \geq 2, \quad \text{and} \quad \downarrow + *q <^+ 0 \quad \text{for } q \neq 1;$$

the only position with a down that fails to be negative is $\downarrow + *$, and, a remote star being available, the surviving nimber can be kept away from $*$. Both displayed inequalities are routine.

On the other hand, the inequality

$$0 <^+ [a_1, \dots, a_{k-2}, a_{k-1}, a_k] + (a_{k-2} - 1) \cdot \downarrow + \star$$

holds because Left, regardless of whether playing first or second, has a winning strategy by always playing in the second component, as long as it contains at least one down. Left decreases the number of downs by one each time she plays. Therefore, there eventually comes a point when the second component is reduced to star or zero, and the first component still has three or more pieces. Left then wins, since she still controls the parity in the first component: writing r for the coordinate of the third-rightmost piece, which is hers, she wins the strip together with $*q$ when moving first, for every q , and also when moving second provided that $r \geq 2$, or that $r = 1$ and $q \geq 2$. This is checked by a routine induction on r , the base cases being the two three-piece strips of Theorem 3.4.

Since $\downarrow + \star <^+ G + a_{k-2} \cdot \downarrow <^+ \uparrow + \star$, by Theorem 1.1, we have $\text{aw}(G) = a_{k-2}$. $\square$

Suppose Left plays first and the atomic weight of a disjunctive sum of strips is one. Under the misère play convention, Left simply reduces Right's advantage whenever possible, raising the atomic weight to two. Right then reduces it to one, Left restores it to two, and so on. Only when this pattern can no longer be maintained does Left need to take the misère play convention into account. The following theorem shows that Left can always win in these situations, which is precisely what makes ALL-SMALL SHOVE tame. In other words, by Theorem 3.2, the two-ahead rule extends from normal play to misère play.

Theorem 3.6. *ALL-SMALL SHOVE is a tame dicotic ruleset.*

Proof. We should check that ALL-SMALL SHOVE satisfies the conditions of Definition 3.1.

The fact that Right cannot decrease the atomic weight of a position by more than one is a direct consequence of Theorem 3.5. The only way to reduce the atomic weight of a strip is by moving its $(k - 2)$ th piece. However, it is clear that even in this case the atomic weight decreases by only one unit.

On the other hand, there are only four situations in which the atomic weight is greater than or equal to one and, on her turn, Left cannot achieve an atomic weight of at least two. Indeed, Left can increase the total atomic weight only in a strip of negative weight, by pushing a black piece that carries a grey piece off the board, so an exceptional position has none; a spare two-piece strip or a spare black piece gives a weight-preserving move, while a total weight greater than two remains at least two after a decrease of one. Since a strip of weight w has at most $w - 1$ pieces to the left of its $(k - 2)$ th piece, a weight-one strip is $[1, a_2, a_3]$, and a weight-two strip is $[2, a_2, a_3]$ or $[\bar{1}, 2, a_3, a_4]$; hence the situations are the following:

- (i) $[1, a_2, a_3] + n \cdot *$, where the stars are strips with two pieces (Theorem 3.3);
- (ii) $[2, a_2, a_3]$;
- (iii) $[1, a_2, a_3] + [1, b_2, b_3]$;
- (iv) $[\bar{1}, 2, a_3, a_4]$.

Under the misère play convention, Left wins in all four situations. In case (i), she moves to an odd number of stars. By Theorem 3.3, she can do this by pushing either the leftmost or the rightmost black piece in the first

component. In case (ii), she plays the rightmost black piece when a_2 is even and the leftmost one when a_2 is odd, reaching $\uparrow + *$ by Theorem 3.4. In case (iii), the first component is $\uparrow$ or $\uparrow + *$. Thus, Left only needs to use Theorem 3.3 to obtain $*$ or 0 in the second component. If the first component is $\uparrow$, she should play to $*$ in the second; otherwise, if the first component is $\uparrow + *$, she should play to 0 in the second. In either case, Left moves to $\uparrow + *$ and wins. Finally, in case (iv), if a_3 is odd, Left plays the black piece in position 2, and if a_3 is even, she plays the rightmost black piece. By Theorem 3.4, this allows her to reach $\uparrow + *$ and win. $\square$

3.2. Case study 2: FLOWER GARDENS

The ruleset FLOWER GARDENS is a dicotic version of BLUE-RED-GREEN HACKENBUSH. Instead of the traditional colours blue, red, and green, we use **b**Lack (**L**eft), **g**Rey (**R**ight), and dash**E**d (**E**ither). In the traditional version, both players may remove green edges; here, these edges are represented by dashed lines.

The components are *flowers*. A flower has a *stem* of dashed edges (at least one), supporting a *blossom* of black and grey edges. By the Colon Principle (Berlekamp et al., 2001), a flower with l black petals and r grey petals has the same value as an equally long one with $l - r$ black petals or $r - l$ grey petals, depending on whether $l \geq r$ or $r \geq l$. For this reason, we assume from now on that the blossoms are monochromatic.

Figure 6 shows petunias (black) and magnolias (grey).⁴ Thus, the flowers have the forms $*n:k$ or $*n:\bar{k}$, where $k \geq 0$ and $k, \bar{k}$, and $*n$ are in canonical forms. *Hackenflowers* are flowers of this type with only one petal, i.e. k is equal to 1 or $\bar{1}$ (on the right in Figure 6). These are the flowers of the misère HACKENBUSH literature (Lo, 2012; Rubinstein-Salzedo and Zhou, 2026); the closely related *sprigs*, in which a single dashed edge supports a string of black and grey edges, were analysed by McKay et al. (2016).

It is known that the atomic weight of a flower depends only on the colour of its petals (Berlekamp et al., 2001). Each petunia has atomic weight equal to 1 and each magnolia has atomic weight equal to -1 . Since, during the course of a game, flowers are deflowered, at some point some stems without

⁴In the classical version, delphiniums (blue) and geraniums (red) are the flowers that are usually mentioned. Since this version is black and white, the flowers are the *black velvet petunia*, which has beautiful black petals, and the *star magnolia*, which has white petals, sometimes light grey.

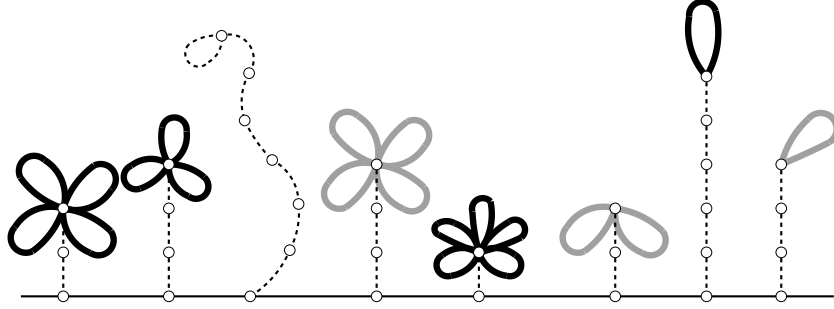


Figure 6: A flower garden.

petals may remain. In those cases, the atomic weight is zero, since they are simply nimbers.

Often, atomic weight theory clarifies that the play is essentially a race. In the example in Figure 6, if Left starts, she removes a magnolia, gaining a lead of two relative to the atomic weight. If Right responds by removing a petunia, his move reduces the difference to one. But again, Left can restore the advantage by removing another magnolia. Eventually, it will be Left's turn in a position where the only flower with petals remaining is a petunia. At that point, it is not difficult to find a winning move. Even if, in the position with a single petunia, it were Right to move, as long as a tall stem (a remote star) is present, Right would not be able to win, since he could not remove the petunia in a way resulting in a $\mathcal{P}$ -position.

Sums of flowers have in fact been analysed in much finer detail than tameness requires: Lo (2012) determines the misère outcome of an arbitrary sum of them, and Rubinstein-Salzedo and Zhou (2026) extends that analysis. Our concern here is only to isolate the boundary positions demanded by Definition 3.1.

Theorem 3.7. *FLOWER GARDENS is a tame dicotic ruleset.*

Proof. There are only two situations in which the atomic weight is greater than or equal to one and, on her turn, Left cannot achieve an atomic weight of at least two. Indeed, a magnolia gives Left a move that increases the total atomic weight by one, so an exceptional position has none, and its weight is the number of petunias; an extra petal or a bare stem gives a weight-preserving move, and a weight greater than two is still at least two after a decrease of one. Hence the two situations are the following:

- (i) There is only one flower with petals in the disjunctive sum, and it is a petunia. Apart from that, there may be some stems without petals.
- (ii) The disjunctive sum consists of exactly two petunias, and each of them has exactly one black petal.

In either case, ignore the petals and regard the dashed edges as a *misère* NIM position. If this position is in $\mathcal{N}$, Left makes a winning dashed-edge move; if it is in $\mathcal{P}$, she removes a black petal and hands the unchanged impartial position to Right. His reply removes a dashed edge and leaves one of the same two cases with fewer edges. Induction therefore gives Left a win. Finally, Right can reduce the atomic weight by at most one, and the argument for grey petals is symmetric. $\square$

3.3. Case study 3: POISONED HACKENBUSH

Our last ruleset is not tame, although a large, easily described subclass is.

Inspired by the poisoned square of CHOMP, the components of POISONED HACKENBUSH are HACKENBUSH-like. The edges standing on the ground form a monochromatic string of *poisoned* edges, above which any configuration of dashed edges may be attached. In a black-based component, Left may remove any edge, whereas Right may remove a poisoned edge only after all dashed edges have gone. The roles are reversed in a grey-based component. As usual, removing an edge brings down everything it disconnects from the ground. Figure 7 shows a disjunctive sum.

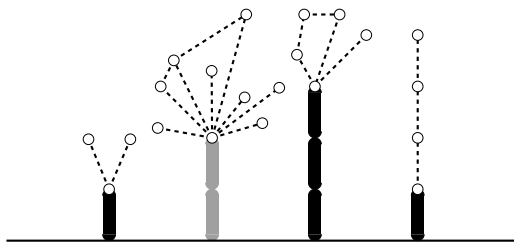


Figure 7: Black- and grey-based POISONED HACKENBUSH components.

This is reminiscent of (but distinct from) Berlekamp’s YELLOW-BROWN HACKENBUSH (Berlekamp, 2009), another all-small variant of HACKENBUSH, which forbids moves on monochromatic components and forces players to cut

their highest edge (in the unrestricted variant, a player can also cut their lowest edge, thereby ending the game immediately).

For a component with dashed edges, atomic weight counts how many moves the disadvantaged player needs before only poison remains.

Theorem 3.8. *If G is a black-based POISONED HACKENBUSH component, then $\text{aw}(G)$ is the number of dashed edges meeting the topmost poisoned edge. The symmetric result holds for grey-based components.*

Proof. Write d for the number of dashed edges meeting the topmost poisoned edge, and k for the number of poisoned edges. Removing one of those d edges brings down the branch above it and lowers the count by one; removing any other dashed edge leaves the count unchanged. Once all d have gone, the component is the nimber $*k$. As in Theorem 3.5, it suffices to verify that

$$G + (d + 1) \cdot \downarrow + \star <^+ 0 <^+ G + (d - 1) \cdot \downarrow + \star.$$

For the first inequality, Right removes an incident dashed edge at each turn; a move of Left's either lowers the count as well or leaves it unchanged, so a down still survives when the poison is exposed. For the second, Left first plays in the $d - 1$ downs, so an incident dashed edge remains when they are exhausted. The resulting endgames are the same sums of downs and nimbers as in Theorem 3.5. Thus $\downarrow + \star <^+ G + d \cdot \downarrow <^+ \uparrow + \star$, and Theorem 1.1 gives $\text{aw}(G) = d$. $\square$

Note that, unlike a flower, whose atomic weight is -1 , 0 , or 1 , a POISONED HACKENBUSH component can have any weight in $\mathbb{Z}$. From left to right, the components of the disjunctive sum in Figure 7 have atomic weights 2 , -8 , 3 , and 1 . The total is -2 , so, by the two-ahead rule, Right wins under the normal play convention: Left takes far too long to reduce the second component to its poison.

The full ruleset is not tame. The leftmost component of Figure 7 has a single poisoned edge meeting two dashed edges, and hence weight 2 . Left can move only to $\wedge$, of weight 1 , or to 0 . Both are $\mathcal{N}$ -positions, so property (ii) of Definition 3.1 fails.

This is the only obstruction. Consider the sums in which no component consists of a single poisoned edge meeting two or more dashed edges; this class is closed under play, since removing a poisoned edge brings the whole dashed part down with it. In any black-based component with dashed edges,

Left now has moves to both 0 and $*$: if it has at least two poisoned edges, she removes the lowest, or the second lowest; if it has only one, she removes the poisoned edge, or the unique incident dashed edge, which brings the whole dashed part down. Right can lower the weight by at most one. The same parity response as in the endgames of Theorem 3.6 handles the exceptional positions, and the symmetric argument applies to grey-based components. Thus, this closed subclass is tame.

More generally, what happens in POISONED HACKENBUSH also occurs in many other wild rulesets: they may contain large classes of tame positions, and Definition 3.1 is local enough to locate them.

4. Final remarks on the full misère structure

In the previous sections, we saw that some dicotic games are infinitesimal with respect to other dicotic games (modulo $\mathcal{D}$). For instance, λ^2 is infinitesimal with respect to λ . We now turn to the existence of infinitesimals in $\mathcal{M}$, showing that none exist. Once again, for ease of notation, we write $\geq$ and $\equiv$ instead of $\geq_{\mathcal{M}}$ and $\equiv_{\mathcal{M}}$ when working with the full misère universe.

Definition 4.1 (cf. (Larsson et al., 2016, Observation 17)). A partially ordered monoid Λ is *Archimedean* if, for every $a, b > 0$, there is a positive integer n such that $n \cdot a > b$.

It is clear that, if a partially ordered monoid Λ is Archimedean, then it cannot contain infinitesimals. However, the converse does not hold: a partially ordered monoid may fail to be Archimedean without containing any infinitesimals. We will show that $\mathcal{M}$ is such an example.

Theorem 4.2. *$\mathcal{M}$ is not Archimedean.*

Proof. Let $H = \{\cdot \mid *, *2\}$ and $G = \{\cdot \mid *\}$. By Theorem 1.4, we know that $0 < H < G$. What we do next is prove that $n \cdot H \not\geq G$, independently of the number of copies of H . To this end, consider the distinguishing game $X = 1$, i.e. one move for Left, given by $\{0 \mid \cdot\}$.

Clearly, Right loses when playing first in $G + X$, since his only option is $* + 1$, to which Left replies with $*$ and wins.

On the other hand, when playing first, Right wins $H + X$. He only needs to move to $*2 + 1$, against which Left has no satisfactory response.

It can also be verified that Right wins $n \cdot H + X$, regardless of the number of copies of H . We argue by induction on n , the case $n = 1$ being the one just

settled. For $n > 1$, Right should move to $(n-1) \cdot H + *2 + X$. If Left responds with $(n-1) \cdot H + X$, then Right replies with $(n-2) \cdot H + *2 + X$, which is the same position with one copy of H fewer, so the induction hypothesis applies. If Left replies with $(n-1) \cdot H + * + X$, then Right continues with $(n-1) \cdot H + X$, forcing Left to move to $(n-1) \cdot H$. After this sequence of moves, Right plays to $(n-2) \cdot H + *$, and repeats the move to $*$ until the end of the game. If Left responds with $(n-1) \cdot H + *2$, then Right continues with $(n-1) \cdot H + *$, and wins as in the previous case.

Thus, when Right plays first, he loses $G + X$ and wins $n \cdot H + X$, which allows us to conclude that $n \cdot H \not\geq G$. $\square$

To prove that $\mathcal{M}$ contains no infinitesimals, we first introduce a preliminary concept. Intuitively, a *Left sprint* of length k should be thought of as the game where Left has to play k moves to reach a negative integer, and Right is unable to make any moves until then. If we wish to specify the negative integer, say $\bar{n}$, then we denote the sprint game by $S_{k,\bar{n}}$. The formal definition is as follows and the concept is illustrated in Figure 8.

Definition 4.3. A *Left sprint* of length $k \geq 0$ and weight $n > 0$ is defined recursively by:

- (i) $S_{0,\bar{n}} := \bar{n}$; and
- (ii) $S_{k,\bar{n}} := \{S_{k-1,\bar{n}} \mid \cdot\}$ if $k > 0$.

The definition of Right sprint is entirely analogous.

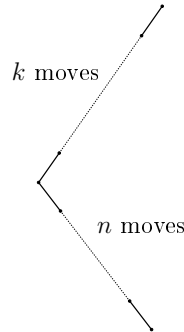


Figure 8: Left sprint of length k and weight n .

Theorem 4.4. $\mathcal{M}$ contains no infinitesimals.

Proof. Consider $0 < H < G$. By Theorem 1.4, $H^{\mathcal{L}} = \emptyset$ while $H^{\mathcal{R}} \neq \emptyset$. Choose $k > \text{rank}(G) + 2$, then $n > k$, and then $m > \text{rank}(n \cdot H) + 2$, and take $X = S_{k, \overline{m}}$.

Playing first, Left loses $G + X$. She has no way to hurry: whenever she steps down the sprint, Right can only reply in G , and since the sprint has length $k > \text{rank}(G)$, the component G is exhausted before she reaches $\overline{m}$. Right is then the first player unable to move, and so wins under misère play; moving in G herself only exhausts it sooner.

Playing first, Left wins $n \cdot H + X$. Now the position is reversed: since $H^{\mathcal{L}} = \emptyset$, Left must step down the sprint, while Right, idle there, is forced to spend his moves in $n \cdot H$. He makes $k - 1$ of them before she reaches $\overline{m}$, and $n > k$ leaves a copy of H still standing, so she does reach it. From $\overline{m}$ Right draws m further moves, whereas at most $\text{rank}(n \cdot H)$ remain anywhere else; as $m > \text{rank}(n \cdot H) + 2$, it is Left who is first unable to move, and so she wins.

Thus Left loses $G + X$ but wins $n \cdot H + X$, whence $G \not\preceq n \cdot H$. As $0 < H < G$ was arbitrary, $\mathcal{M}$ contains no infinitesimals. $\square$

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