

How mathematical games shape pre-service teachers' generalization processes: a qualitative case study

Cómo los juegos matemáticos configuran los procesos de generalización de los docentes en formación: un estudio de caso cualitativo

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Abstract

This study investigates how pre-service mathematics teachers engage in mathematical generalization within game-based learning environments. Although game-based learning has gained increasing attention in mathematics education, existing research has largely emphasized engagement, motivation, and achievement, with comparatively limited focus on how such environments support the development of mathematical generalization processes. In particular, little is known about how pre-service teachers transition from concrete gameplay experiences to symbolic reasoning and formal mathematical generalizations, or how these processes relate to justification and proof. Addressing this gap, the study employed a qualitative multiple case study design involving 28 pre-service mathematics teachers enrolled in a Teaching Mathematics with Games course at a public university. Grounded in constructivist and problem-solving perspectives, the study examined how participants' actions, representations, and discussions during gameplay contributed to emerging generalizations. Data were collected through classroom observations, audio-recorded whole-class and small-group discussions, participants' written artifacts produced during gameplay, and re-

searcher field notes. Participants played mathematical games, documented observations, identified patterns, and attempted to formulate and justify generalizations. The data were analyzed using qualitative analysis techniques supported by illustrative excerpts from participants' discourse and written work. Findings indicate that most participants actively generated predictions, drew inferences, and relied on intuitive reasoning during gameplay. However, many experienced difficulties in constructing valid and general mathematical statements and in justifying their conclusions through formal reasoning. These results reveal persistent challenges in the interconnected practices of generalization and proof, despite the exploratory and engaging nature of game-based environments. The study highlights the need for structured pedagogical support to help pre-service teachers transform exploratory insights into mathematically justified generalizations.

Keywords: Mathematics education, Mathematical games, Mathematical generalization, Pre-service teachers, Case study

Resumen

Este estudio analiza cómo los docentes en formación de matemáticas participan en procesos de generalización matemática en entornos de aprendizaje basados en juegos. Aunque este enfoque ha ganado atención en la educación matemática, la investigación previa se ha centrado principalmente en la motivación, la participación y el rendimiento, mientras que ha prestado menos atención al desarrollo de la generalización matemática. En particular, sigue siendo limitada la comprensión de cómo los docentes en formación transitan de experiencias concretas de juego hacia el razonamiento simbólico y las generalizaciones formales, así como de la relación de estos procesos con la justificación y la demostración.

Para abordar esta laguna, se adoptó un diseño cualitativo de estudio de casos múltiples con 28 docentes en formación matriculados en un curso de Enseñanza de las Matemáticas con Juegos en una universidad pública. Desde una perspectiva constructivista y de resolución de problemas, se analizó cómo las acciones, representaciones y discusiones durante el juego contribuyen a la construcción de generalizaciones. Los datos se recopilaron mediante observaciones en el aula, grabaciones de discusiones en grupo, producciones escritas y notas de campo. Los participantes jugaron, registraron observaciones, identificaron patrones e intentaron formular y justificar generalizaciones. El análisis cualitativo se apoyó en ejemplos del discurso y de los trabajos escritos. Los resultados muestran que la mayoría de los participantes generaron predicciones e inferencias basadas en razonamiento intuitivo. Sin embargo, muchos tuvieron dificultades para formular enunciados matemáticos generales y justificar formalmente sus conclusiones. Estos hallazgos evidencian desafíos persistentes en la relación entre generalización y demostración y destacan la necesidad de un apoyo pedagógico estructurado para transformar ideas exploratorias en generalizaciones matemáticamente justificadas.

Palabras clave: Educación matemática, Juegos matemáticos, Generalización matemática, Docentes en formación, Estudio de caso

Introduction

Generalization, the process of discerning overarching patterns, structures, and relationships from specific instances, lies at the heart of mathematical reasoning and knowledge development (Tall et al., 2012; Mason et al., 2010). It extends beyond simple pattern recognition to include formulating statements that hold beyond observed cases and justifying their general validity, forming a foundation for algebraic thinking and proof (Carpenter & Franke, 2001; Kieran, 2004; Stylianides, 2007). As a cornerstone of mathematical proficiency, generalization is emphasized in curriculum standards (National Council of Teachers of Mathematics [NCTM], 2000) and requires sophisticated cognitive engagement. Mason (2010) underscored its centrality, arguing that mathematical thinking depends on recognizing and cultivating generalization. Understanding how learners particularly future mathematics teachers develop generalization competencies is therefore a key concern in mathematics education research.

Pre-service teachers (PTs), as future facilitators of mathematical reasoning, must themselves grapple with generalization and proof. Research shows both their potential and persistent challenges in these areas. Studies distinguish between expressive generalization (constructing general claims) and receptive generalization (recognizing given generalizations) and emphasize the dynamic nature of generalizing activity (Ellis, 2007; Ellis et al., 2012). Other research documents PTs' reliance on empirical arguments and their difficulties transitioning to deductive reasoning (Stylianides, 2007; Weber, 2001). Engaging PTs in rich tasks that demand generalization and justification is considered essential for developing pedagogical content knowledge (Herbst & Chazan, 2011; Zaslavsky, 2005).

While these studies provide insights into generalization in traditional tasks, less attention has been given to how generalization unfolds in alternative learning environments. One such context is mathematical gameplay, where games are known to enhance engagement, reduce anxiety, and promote positive attitudes toward mathematics (Cai et al., 2006; Ernest, 1986; Lester et al., 2014). Beyond affective benefits, games offer distinctive epistemic affordances: mathematical structures are embedded in rule-based systems requiring strategic reasoning and prediction (Juul, 2005; Koparan, 2019; Koparan, 2021; Koparan, 2022).

Gameplay can support embodied exploration (Sinclair et al., 2013), low-risk conjecturing and identification of structural relationships (Ellis et al., 2012; Herbst & Chazan, 2011). Iterative feedback cycles may further promote productive struggle and refinement of ideas (NCTM, 2000; Perrotta et al., 2013).

Despite these affordances, empirical research on how games support PTs' processes of generalization and justification remains limited. Existing studies often emphasize motivation or achievement, while the cognitive and epistemological dimensions of generalization in game-based contexts particularly for those learning to teach mathematics remain underexplored. This gap is important because PTs' experiences with reasoning shape their future instructional practices (Zaslavsky, 2005).

The present study addresses this gap by examining how pre-service middle school mathematics teachers develop mathematical generalizations while engaging in strategic mathematical games. Rather than only documenting that generalizations occur, the study investigates their nature, the processes through which they emerge, and how features of games may support or constrain generalization and justification. The guiding research question is: How does engagement in strategic mathematical games shape pre-service teachers' processes of mathematical generalization and their approaches to developing and validating general claims?

Purpose and research question

This study aims to examine how pre-service teachers develop mathematical generalizations while engaging in mathematical games. Specifically, it seeks to understand the processes, representations, and reasoning strategies involved in generalizing within game-based contexts. The main research question is: How does the process of deriving mathematical generalizations unfold as pre-service teachers participate in mathematical games?

To address this question, the study explores the following sub-questions:

- 1) How do pre-service teachers arrive at generalizations while playing mathematical games?
- 2) What types of generalizations and strategies do pre-service teachers use in these contexts?
- 3) What forms of representation do pre-service teachers employ when expressing generalizations?
- 4) What difficulties do pre-service teachers encounter during the process of generalizing?

By focusing on the emergence of mathematical generalization within gameplay, this study contributes a process-oriented perspective to the literature and sheds light on how game-based environments can support or limit the development of generalization and justification among future mathematics teachers.

Method

This section outlines the methodological framework of the study, including the research design, participant characteristics and research context, theoretical framework, data collection procedures, and data analysis process. A qualitative multiple case study approach was adopted to investigate how mathematical generalization processes emerged within game-based learning environments.

Research Design

This study employed a qualitative multiple case study design (Stake, 2006; Yin, 2018) to explore how mathematical generalization processes emerged in game-based learning environments among pre-service middle school mathematics teachers. A case study approach was considered appropriate because it allows for an in-depth examination of complex cognitive processes within their natural instructional setting. Each game-based instructional implementation was treated as a bounded case, enabling an examination of participants' generalization processes across different mathematical contexts. The study was grounded in an interpretive qualitative paradigm that seeks to understand how participants construct meaning through interaction, discussion, and reflection. This design enabled the researchers to capture the dynamic, socially mediated, and context-dependent nature of mathematical generalization as it unfolded during gameplay.

Participants

The participants consisted of 28 pre-service middle school mathematics teachers (17 female, 11 male; mean age = 19.72) enrolled in an undergraduate mathematics teacher education program at a public university. At the time of the study, they were in the second year of their four-year program and were taking the elective course Teaching Mathematics Through Games during the Fall semester of the 2022–2023 academic year. Participants were selected using purposeful sampling (Patton, 2015), specifically through a typical case sampling strategy (Gerring, 2007). The study was conducted at the faculty of education where the researchers were employed, which facilitated access and sustained classroom observation. This context was considered appropriate because it provided an authentic instructional setting in which pre-service teachers engaged systematically with mathematical games as part of their formal teacher education program. The inclusion criteria required participants to (a) be officially enrolled in the pre-service middle school mathematics

teacher education program, (b) be registered for the Teaching Mathematics Through Games course during the implementation semester, (c) not have previously completed this course, and (d) sign a voluntary participation consent form. All preservice teachers enrolled in the course met these criteria and were therefore included in the study. Rather than aiming for statistical generalizability, the study sought an in-depth and contextualized understanding of participants' reasoning processes; thus, purposeful sampling aligned with the interpretive goals of qualitative case study research. The course structure emphasized collaborative problem solving, discussion, and reflection, creating a socially interactive learning environment. These peer interactions provided opportunities for shared meaning-making and supported the development of mathematical generalizations through dialogue and collective reasoning, consistent with sociocultural perspectives on learning (Vygotsky, 1978).

Theoretical Framework

The study was guided by an integrated theoretical framework that informed the design of the learning environment, the development of data collection tools, and the analytical lens used to interpret participants' mathematical generalization processes. Three complementary perspectives were operationalized in interconnected ways. Bruner's (1961) enactive–iconic–symbolic representation stages informed the instructional design and analytical coding of representations. The selected games required participants to first engage in enactive interaction through physical or rule-based gameplay, then transition to iconic representations such as diagrams, sketches, or visual pattern tracking, and finally express their reasoning through symbolic generalizations (e.g., algebraic rules or verbal general statements). This progression guided both the structure of classroom prompts (e.g., encouraging students to “show,” “draw,” and “express generally”) and the analysis of written artifacts and discussions, where representations were coded according to these representational levels. Vygotsky's (1978) social constructivist perspective shaped the design of classroom interactions and the interpretation of discourse data. Tasks were intentionally structured to include individual exploration followed by whole-class discussions, allowing participants to refine and formalize their ideas through dialogue. During data analysis, audio-recorded discussions were examined to identify moments of socially mediated meaning-making, such as when participants built on peers' ideas, negotiated conjectures, or collectively reformulated informal observations into more general statements. This perspective framed generalization as a socially situated process rather than solely an individual cognitive act.

Polya's (1945) problem-solving cycle informed both the design of game-based tasks and the analysis of reasoning processes. The games were selected because they naturally required participants to understand a problem situation, explore strategies, test emerging conjectures, and refine their reasoning—mirroring Polya's stages of problem solving. In the analysis phase, participants' solution paths were examined for evidence of these stages, particularly the transition from exploratory reasoning to the formulation of generalized rules and attempts at justification. Together, these frameworks functioned not as abstract theoretical references but as practical lenses that guided task design, shaped data collection (e.g., prompts, discussion structure, artifact collection), and structured the analysis of representations, discourse, and problem-solving processes. This integrated approach allowed the study to capture mathematical generalization as a dynamic process emerging through action, interaction, and reflective formalization within game-based learning contexts.

Data Collection

Data collection was conducted during the Fall semester of the 2022–2023 academic year and spanned approximately 14 weeks, corresponding to the full duration of the Teaching Mathematics Through Games course. Data were gathered during regularly scheduled course sessions in which game-based mathematical activities were implemented. This extended time frame enabled the researchers to trace the development of participants' generalization processes across multiple instructional cycles and collaborative learning experiences.

The instructional design incorporated four games: Jumping Frogs Game, Sliding Puzzle, Diagonal Game, and Pizza Cutting Game each selected for its potential to naturally elicit mathematical generalization. Selection criteria included opportunities for pattern recognition, the use of multiple representations, and the promotion of social interaction. A summary of these games, including their structures and generalization goals, is presented in Table I. Photographs illustrating the learning environment and participants' engagement in game-based activities are provided in Chart I for contextual purposes; these visuals serve an illustrative rather than analytic function. Data collection followed Cobb's (2000) three-stage cyclical model: design and planning, implementation, and retrospective analysis.

Design and Planning.

During this stage, the researchers evaluated each game for its alignment with preservice teachers' cognitive structures and its potential to pro-

voke processes of noticing patterns, forming conjectures, and expressing generalizations. Instructional prompts and discussion questions were prepared to support the emergence of mathematical reasoning during gameplay.

Implementation.

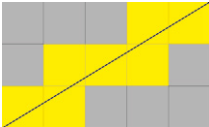
The course spanned 14 weeks; however, the four focal games analyzed in this study were implemented during four dedicated 90-minute sessions. In each session, participants first engaged in individual work (approximately 15–20 minutes per task), followed by whole-class discussions in which emerging generalizations were shared, compared, and refined. Discussions focused on reasoning strategies, representational forms (e.g., diagrams, symbolic expressions, verbal rules), and epistemological obstacles encountered during problem solving. The first author was also the course instructor and therefore assumed a dual role as instructor and researcher. During the sessions, the researchers facilitated the learning environment, provided materials for gameplay, and posed guiding questions intended to help participants reveal underlying mathematical structures and articulate generalizations. To reduce potential researcher bias associated with this dual role, the second author independently reviewed the data during the analysis phase, and analytic decisions were discussed collaboratively. Multiple sources of qualitative data were collected to ensure trustworthiness and methodological rigor through triangulation. These sources included audio recordings of whole-class discussions and small-group interactions, which captured participants' verbal reasoning and collaborative meaning-making processes. In addition, written artifacts produced during gameplay such as strategy notes, sketches, symbolic representations, and written generalizations provided insight into how participants externalized and refined their mathematical thinking. Researcher field notes further documented classroom observations, interaction patterns, and notable episodes of mathematical reasoning, contributing contextual depth and supporting the interpretation of the data. In total, approximately four hours of audio recordings and over 112 pages of written student artifacts were collected.

Retrospective Analysis.

Following each instructional session, the research team reviewed field notes and audio recordings to identify significant episodes of mathematical reasoning and emerging generalization attempts. These reflections informed minor adjustments to subsequent tasks and discussion prompts, consistent with the iterative nature of Cobb's (2000) design framework.

All audio recordings were transcribed verbatim. Written artifacts were collected and organized alongside field notes to prepare a comprehensive dataset for the formal qualitative analysis described in the Data Analysis section.

Table I. Some of the Games Used to Generalise

Game	Name of the Game /How it is Played / The Purpose and Rule of the Game
	
1	Jumping Frogs Game (Individual - Whole class)) This algorithm game aims to replace the frogs on the left with the frogs on the right. Frogs can move in the direction they look. This movement is of two types. If the stone in front of the frog is empty, the frog jumps to this stone. If the stone in front of the clicked frog is full but the next stone is empty, it can jump over the frog in front of it. In at least how many moves can this goal be achieved, as the numbers of frogs on the left and right are n and m, respectively? (Link: https://nrich.maths.org/content/00/12/game1/frogs/index.html#/student/2/2)
2	Sliding Puzzle (Individual- Whole class) In the 4×4 type structure, the squares can move up, down, right and left. In how many moves can the green piece be moved to the yellow empty part? Can you guess what will be the minimum number of moves in any $n \times n$-type structure? If your guess is correct, can you explain why? Can you guess what the minimum number of moves will be in any structure of type $m \times n$ with $m > n$? Can you explain why? (Link: https://nrich.maths.org/games/sliding-puzzle)
3	Diagonal Game (Individual - Whole class)  Consider a rectangle of unit squares of type 5×3. Draw the diagonal of the rectangle. "Can you determine the connection between the number of unit squares the diagonal passes through and the side lengths of the rectangle?" If your guess is correct, can you generalise it for rectangles of type $m \times n$?
4	Pizza Cutting Game (Individual - Whole class) How many pieces can a pizza be cut with 5 knife cuts? Can you find the maximum number of pieces the pizza will be cut into when the knife cuts n? (Parts don't have to be equal).

In Chart I several sections from the game-based learning environment are shown.

Chart I. Some sections from the learning environment



The course sessions took place in a classroom arranged with round tables to facilitate face-to-face interaction and collaborative gameplay. Ethical approval for the study was obtained from the relevant institutional ethics committee prior to data collection, and all participants provided informed consent. This research was reviewed and approved by the Human Research Ethics Committee of Zonguldak Bulent Ecevit University. Approval Code: 96985. Approval Date: 04.11.2021.

Data Analysis

This study employed a qualitative descriptive-exploratory research design. Data analysis was conducted as an iterative and multi-layered process combining continuous and retrospective analysis. Continuous analysis took place throughout the instructional process. After each lesson, the instructor-researcher recorded detailed reflections and observational notes regarding classroom interactions, students' strategic moves, emerging patterns of reasoning, and moments of mathematical generalization observed during gameplay. This ongoing analytic process informed subsequent instructional decisions and allowed the researchers to capture learning trajectories as they unfolded. Retrospective analysis was conducted after the completion of the instructional sequence. At this stage, the entire dataset including audio recordings, written artifacts, and field notes was re-examined holistically. Students' written responses, diagrams, symbolic representations, and strategic notes were systematically reviewed and compiled into an integrated dataset for in-depth analysis. The analysis focused on preservice teachers' generalization process-

es within game-based learning environments. An analytical framework based on the actions and indicators presented in Table II guided this process, allowing the researchers to trace how participants progressed from understanding the game context to producing generalizations and justifications.

Table II. Actions and indicators in the generalization process to games

Actions	Indicators
Understanding the game	Identifying how the game works; determining variables; examining initial cases or specific situations
Observation	Organizing outputs using lists, tables, figures, graphs, or symbolic representations
Strategy development	Considering unknown future steps by analyzing patterns in outcomes
Prediction and inference	Making conjectures about all possible cases based on experimental results, though not yet formally justified
Test–verify–refute	Generating cases to test predictions and inferences; evaluating correctness (often limited to specific cases rather than general proof)
Generalization	Formulating statements that are claimed to hold in general
Proof	Justifying or proving a generalization using logical reasoning

Data analysis followed a hybrid inductive–deductive coding approach (Timmermans & Tavory, 2012). In the first phase, deductive theoretical codes derived from Bruner’s representational stages, Vygotsky’s social constructivist perspective, and Polya’s problem-solving cycle were used to frame how students’ actions, interactions, and symbolic expressions reflected stages of mathematical generalization. In the second phase, inductive coding was employed to identify emergent, game-specific triggers (e.g., breakdown of a strategy, loss of a turn, or unexpected outcomes) that prompted shifts in reasoning. During this phase, students’ justification patterns were also examined in relation to established proof and validation frameworks in mathematics education literature. In the third phase, a microgenetic analysis (Siegler, 2006) traced how individual and group generalization processes evolved across successive game sessions, allowing the researchers to document conceptual change over time.

Coding was conducted by two researchers. The first researcher is a specialist in mathematics education and led the instructional implementation, while the second researcher, a physicist and teacher educator in science education, participated independently in the coding and analysis stages. Rather than calculating inter-coder agreement coefficients, the

study employed a consensus coding approach. Both researchers initially coded the data separately and then engaged in systematic comparison meetings to discuss discrepancies, negotiate interpretations, and reach agreement on final coding decisions. This process strengthened analytic depth and reduced individual interpretive bias.

During gameplay sessions, both classroom discussions and participants' written explanations were coded in alignment with the action-indicator framework presented in Table II. In this way, verbal reasoning, peer interactions, and individually constructed written generalizations were examined through the same analytical structure, allowing for consistent comparison across data sources. In addition, participants' representations were systematically classified according to their form, including verbal descriptions, numerical tables, diagrams, graphical models, and symbolic expressions. This representational analysis constituted an additional analytic layer that complemented the action-based coding framework. It enabled the researchers to examine how different representational modes supported, transformed, or constrained the development of mathematical generalizations.

To enhance trustworthiness and methodological rigor, multiple validation strategies were employed. Participant validation (member checking) occurred during whole-class discussions, where emerging interpretations, conjectures, and generalization attempts were revisited and clarified with participants. In addition, data triangulation across audio recordings, written artifacts, and field notes supported the consistency and credibility of interpretations. Discrepant cases were deliberately examined to refine emerging explanations.

In reporting the findings, the researcher is labeled as R, and participants are coded as PT1–PT28. Direct quotations, student-generated representations, and written work samples are included to provide transparency and allow readers to trace the connection between raw data and interpretations (Miles & Huberman, 1994).

Results

The findings derived from the research data are presented as responses to the sub-research questions.

Findings for the First Sub-Problem

The first sub-problem of the research is: *“How do pre-service teachers reach generalizations while playing the games?”* The findings related to this question are presented through direct quotations from the classroom

discussions about each game. Excerpts from the class discussion on the *jumping frogs* game are shown in Table III.

Table III. Sections from the reasoning of the pre-service teachers about the jumping frogs game

R/PT	Class discussions for the jumping frogs game	Actions
R	In how many moves can the frogs change places? Do you have a suggestion for a solution?	Strategy development
PT2	I thought that if we create a position so that the frogs can jump, the number of moves will decrease.	
R	So, how many frogs are on the right and left?	
PT2	Let's start with at least the frog. Have a frog on each side.	Simplification
R	What would be the minimum number of moves in this case?	Observation
PT2	3 moves (Showing)	
R	What would your answer be if there were two frogs on each side?	
PT2	8 moves (Showing)	Observation
R	What would your answer be if there were three frogs on each side?	Prediction-Inference
PT2	I found 17 but not sure	
PT5	I didn't find	
R	"What observations did you make while playing the game? What are your thoughts about the game?"	Prediction-Inference
PT14	It doesn't matter which way we start. Because it's symmetrical.	
PT25	If frogs of the same colour are consecutive, the game is locked.	
PT17	We need to create a gap between frogs of the same colour so that other frogs can jump through. This ensures fewer moves	Prediction-Inference
PT9	I found 15 (Showing)	Observation
PT2	Returns the number of moves , where n is the number of frogs.	Generalization
R	Is this generalization always true?	Verification
PT2	Yes, if there are four frogs on each side, it is 24.	
R	Well, if the numbers of frogs on the right and left are different, what can you say if, for example, there are 3 frogs on the left and 2 frogs on the right?	
PT2	I can try. 11 moves. (Trying and showing)	Testing-Observation

R/PT	Class discussions for the jumping frogs game	Actions
R	Did the previous generalization work?	
PT2	No, I cannot use it. When the frog numbers are different, I think there is a different generalization or it may be an all-encompassing generalization.	Refute
R	You can also look at other situations where the frog numbers are different. (They're trying)	
PT9	11 moves for 3,2, 19 moves for 4,3, 23 moves for 3,5	Observation
R	Can you generalise by considering all the cases?	
PT5	Number of frogs on the left and right as m and n $(m.n)+(m+n)$	Generalization
R	Is this generalization correct?	
PT2	Yes, I tried. It always works in different situations.	Verification
PT15	It was very nice. All the stones fell into place.	
R	Can you prove this?	

As seen in Table III, pre-service teachers began with intuitive strategies, followed by observations and inferences through experimentation. These led to accurate generalizations; however, no justification or proof was provided.

The dialogue in Table III exemplifies the iterative generalization process triggered by game-based inquiry. PT2's $(n + 1)^2 - 1$ generalization in symmetric cases (empirical validation) collapsed when tested with asymmetric parameters (3 left/2 right). This collapse demonstrates how the dynamic rule flexibility of the game systematically undermines intuitive predictions. At the critical juncture, PT9's data sharing (3.2:11 moves; 4.3:19 moves) feeds into PT5's $mn + m + n$ formula, demonstrating social knowledge construction and the contribution of peer scaffolding to generalization quality. The progression from physical manipulation (showing moves) → numerical recording → symbolic generalization reveals how the action-visual-symbolic stages are embodied in the game context. The researcher's final question (Can you prove it?) reveals a fundamental gap: While participants could provide inductive validation, they had difficulty with deductive justification. This finding offers pedagogical implications for how play experiences can serve as a bridge to formal proof. Related class discussions on the sliding puzzle game are presented in Table IV.

Table IV. Sections from the reasoning of the pre-service teachers about the sliding puzzle game

R/PT	Class discussions for sliding puzzle game	Actions
R	"What are your thoughts on the game?"	
PT8	I think we can make the game simpler.	Simplification
R	How? Can you explain?	
PT8	We can look at simpler cases. Before the 4x4 case, we can look at the 3x3 and 2x2 cases	Strategy development
R	Why do you want to do this?	
PT8	Because there are fewer moves in 2x2 than others. Also, if we can find the number of moves in 2x2 and 3x3 games, we can guess the others.	Prediction
R	So, what is the solution for a 2x2 game?	
PT8	8 (Showing)	Observation
R	So, what can you say about the 3x3 game?	
PT7	We can move the green square along the edge (south-west) (Showing)	Prediction-Inference
R	How many moves?	
PT7	19 moves for 3x3	
PT7	It can be done this way. But this way there are a lot of moves. Leaving the edge was not a short solution	Refute
PT21	I completed it in at least 13 moves	Testing-Observation
R	Can you show me the moves? (Showing)	Verification
PT21	The number of moves in a 4x4 game can be 21	Prediction-Inference
R	What did you think?	
PT21	There are always 8 moves between them. It goes like 5, 13, 21,...	Observation
R	So, do you have a solution suggestion for the 4x4 game?	
PT4	To make the least number of moves, we must move down and to the left, passing through the points on the diagonal. If we do this, there will be 21 moves.	Prediction-Inference
R	Can you show me? (Showing)	Verification
R	Why did you think so?	
PT4	If we could move diagonally, we would go over the diagonal for the fewest moves. Because the shortest distance is the length of the hypotenuse. But since there is no diagonal movement, we must follow the path of the diagonal with down and left movements.	Verification
R	Can you generalise about the game?	
PT13	8n-3	Generalization

R/PT	Class discussions for sliding puzzle game	Actions
PT7	It wouldn't be $8n-3$. n starts from 1, not 2. It's not 5 for $n=2$	Refute
PT10	According to me $8n-11$	Generalization
R	Does it provide in all situations?	
PT10	Yes, I checked. Also, the number of moves is always odd. Because $8n-11$ always produces odd numbers.	Prediction-Inference
R	Well, I have one more question. Why is that? Can you prove this?	
PT10	What proof do you want?	
R	A logical explanation or justification that the generalization is $8n-11$ in an $n \times n$ -type game.	

As seen in Table IV, this dialogue reflects the participants' strategies for simplifying a complex problem (PT8's investigation of $2 \times 2 / 3 \times 3$ instead of 4×4) and the tension between mathematical intuition and formal generalization. PT4's calculation of the shortest path using hypotenuse logic under the "diagonal move constraint" (21 moves) demonstrates how geometric concepts are integrated into game mechanics. PT21's observation of numerical patterns (5,13,21...) and PT10's generalization of $8n-11$ reveal the limits of empirical discovery: Although structural features such as the odd number of moves ($8n-11$, odd), are noticeable, the teacher's demand for proof (Logical justification?) reveals the epistemic gap between inductive observation and deductive verification. PT7's refutation of the $8n-3$ formula by saying "it doesn't give 5 for $n=2$ " is evidence of how peer criticism ensures mathematical certainty. The process in these dialogues reveals the role of games not only in developing strategies but also in building a culture of mathematical verification. Related class discussions appear in Table V.

Table V. Sections from the reasoning of the pre-service teachers about the diagonal game

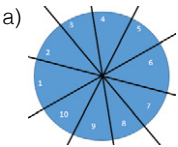
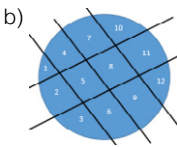
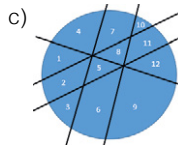
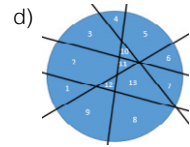
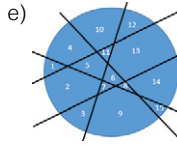
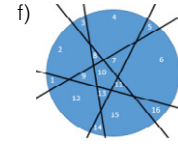
R/PT	Class discussions for the diagonal game	Actions
R	Is there a relationship between the number of squares the diagonal passes through and the sides of the rectangle?	
PT28	I think we can start the review with small structures.	Strategy development
PT19	We can examine it by fixing one edge and changing the other edge.	Strategy development
PT8	In a 5×3 rectangle, the diagonal goes through 7 squares.	Observation
R	Anyone else try it?	
PT15	In 2×1 the diagonal passes through 2 squares, in 5×5 the diagonal passes through 5 squares	Observation

R/PT	Class discussions for the diagonal game	Actions
PT6	In 4x7, the diagonal passed through 10 squares	Observation
PT11	In 4x6, the diagonal passed through 8 squares.	Observation
R	What can you say based on these results?	
PT7	I think it has to do with the side lengths being odd or even	Prediction-Inference
R	How? Can you explain?	
PT7	There are cases where the side lengths are both odd, both are even, one is odd and the other is even. If the sum of these lengths is even, it is necessary to subtract 1, and if it is odd, it is necessary to subtract 2.	Prediction-Inference
PT22	In my opinion this can't be right. In 3x3, the diagonal goes through 3 squares.	Refute
PT21	The larger one if both are even numbers. For example, 4 in 2x4, 6 in 3x6.	Prediction-Inference
PT1	I noticed this. If it is not square, it becomes $m+n-1$.	Prediction-Inference
PT	In 4x6 it should be 8 but 9. Is not correct.	Refute
R	So, the $m+n-1$ is not correct?	
PT21	It is true when the numbers are prime between them.	Prediction-Inference
R	Can you explain?	
PT21	In 2x1, 4x7 and 5x3, the side lengths are co-prime. According to the formula, 2, 10, and 7 are in order. So, the number of squares the diagonal goes through $m+n-1$ is correct.	Verification
R	So what can you say in this case if the side lengths are not prime between them?	
PT7	We subtract the greatest common divisor from the sum of the sides. So $m+n-\text{GCD}(m,n)$	Generalization
R	Can you give an example?	
PT7	For example, in 9x6, $9+6-\text{GCD}(9,6)=9+6-3=12$. It also provides other examples that are not prime among them.	Verification
R	What else can you say?	
PT7	The second formula covers the first case. So $m+n-1$ is a special case in $m+n-\text{GCD}(m,n)$	Generalization
PT10	Very interesting.	
R	So, why is that? Can you prove this mathematically?	

The dialogue in Table V exemplifies the contextual discovery dynamics of mathematical generalization. It was observed that the participants took a strategic approach by starting with simple cases (PT28/PT19's 2x1, 5x3 examination). PT7's "odd-even length" hypothesis was refuted by PT22's 3x3 counterexample, revealing the limits of empirical predictions. The turning point, PT21's discovery of the "relatively prime numbers" condition (for the formula $m + n - 1$), brought the problem to the level of

number theory. The final generalization (PT7's formula $m + n - GCD\ m,n$) represents the algebraic solution of a geometric problem. It can be said that the dialogue includes the skills of mathematical comprehensiveness: realizing that $m + n - 1$ is a special case of $GCD = 1$, and establishing a connection between geometry and number theory. Class discussions on the pizza-cutting game are detailed in Table VI.

Table VI. Sections from the reasoning of the pre-service teachers about the Pizza Cutting Game

R/PT	Class discussions for Pizza Cutting Game	Actions
R	Do you have an estimate of how many pieces the pizza can be divided into with 5 knife cuts?	
PT7	If there are more intersections, there may be more pieces.	Strategy development
PT12	I got 10 pieces.	Observation
PT16	I got 12 pieces.	Observation
R	I also got 12 pieces.	Observation
PT5	13 pieces.	Observation
R	Can you show me how you found it? (They show Figure A, Figure B, Figure C)	
R	Can anyone cut the pizza into more pieces?	
PT20	14 pieces.	Observation
PT13	I got 15 pieces.	Observation
PT7	I got 16 pieces.	Observation
R	Can you show me how you found it? (They show Figure D, Figure E, and Figure F)	
a) b) c) d) e) f)		
R	So, what can you say based on your observations?	
PT5	After the first cut, we can make the second cut in two ways. Either parallel to the first or by cutting through the first. If we want the pieces to be more, each cut should be such that it cuts the previous cuts.	Strategy development

R/PT	Class discussions for Pizza Cutting Game	Actions
R	So, what about the number of parts in these two cases?	
PT5	If they are parallel to each other, it becomes 3 pieces, when they cross each other, it becomes 4 pieces.	Verification
R	So, how can you use this knowledge in the game?	
PT5	We must cut the previous ones in each cut without making parallel cuts	Prediction-Inference
R	How many pieces can be obtained in this way?	
PT5	16 pieces. (Figure F)	Verification
R	So, if the knife stroke is 6, how many pieces will there be? Can you find it?	
PT17	In each segment, the number of pieces increases by 1, 2, 3....	Prediction-Inference
R	How?	
PT17	In other words, it progresses as 1 piece for 0, 2 pieces for 1, 4 pieces for 2, 7 pieces for 3, 11 pieces for 4, and 16 pieces for 5.	Prediction-Inference
R	Can you generalise the maximum number of pieces when the number of cuts is n?	
PT5	$n(n+1)/2+1$	Generalization
R	Are your results compatible with this formula?	
PT5	Yes, it is compatible.	Verification
R	So, why does this generalization work? Can you prove this?	
PT5	The amount of increase in each section is 1, 2, 3, ... Their sum is $n.(n+1)/2$. At first, we got 1 pizza. If we add that, it becomes $n.(n+1)/2+1$.	Proof
PT23	$f(0)=1, f(1)=2, f(2)=4$, where n is the number of cuts, f(n) being the maximum number of parts..The function can be represented iteratively as $f(n) = n + f(n-1)$. If the values $f(n-1), f(n-2), \dots$ are written in this expression, respectively, $f(n)=n+f(n-1)=n+n-1+f(n-2)=n+n-1+n-2+f(n-3)=n+n-1+n-2+n-3+\dots+1+f(0)$ $f(n)=n+n-1+n-2+\dots+1+f(0)$ is obtained. $f(n)=(n.(n+1))/2+1$. With this optimization, the maximum number of parts can be determined for all cuts.	Proof

This episode illustrates how participants engaged in combinatorial reasoning while attempting to maximize the number of regions. While participants initially progress through experimental results (PT12:10, PT16:12, PT17:16), PT5's "no-parallel-cuts" strategy marks the critical breakthrough: Recognizing that maximum pieces occur when new cuts intersect all prior ones. This demonstrates mathematical intuition under geometric cont

straints. PT17’s incremental pattern (1,2,3,4,5…) and PT5’s formula $\frac{n \cdot (n + 1)}{2}$

reflect the evolution from empirical data to algebraic generalization. The uniqueness lies in dual proof approaches: PT5’s inductive proof: Based on summing increments (1+2+3+...+n) PT23’s recursive proof: Derives functional relation $f(n) = n + f(n - 1)$. This duality proves games nurture not just discovery but alternative justification pathways. The drawings (Figures A-F) serve as a bridge in concretizing abstract generalizations. Unlike other games, the pizza-cutting task prompted some attempts at proof. While pre-service teachers engaged in various reasoning processes such as strategy development, inference, and generalization their weakest participation was consistently observed during the proof phase, indicating a need for further support in this area.

Findings for the Second Sub-Problem

The second sub-problem of the research is: “*What are the generalization types and strategies that pre-service teachers use for games?*” The findings related to this question are provided in Table VII.

Table VII. Generalization types and strategies of pre-service teachers towards games

Generalization type	Strategies	Frequency			
		Game 1	Game 2	Game 3	Game 4
Numerical approach	Random trial and error	13	11	15	17
	Systematic trial and error		6	4	
	Create a table				15
	Numerical use of finite differences in a table				7
	Extending the table	3	4	8	23
	Fixing one of the independent variables	2	2	3	
	Generalization as a piecewise function			3	
	Continuing the pattern to a near-finite step	10	9	11	12
	Refutation by contradiction	4	3	9	7

Generalization type	Strategies	Frequency			
		Game 1	Game 2	Game 3	Game 4
Visual approach	Counting one by one	10	15	9	20
	Drawing a figure or diagram	8			28
	Creating simpler structures	7	6	11	20
	Keeping some variables constant	5	4	6	
	Focus on intersections				9
	Leveraging the symmetry feature	4			
	Refutation by contradiction	2	2	4	6

Table VII presents the frequency distribution of generalization strategies **across four mathematical games** (Jumping Frogs, Sliding Puzzle, Diagonal Game, Pizza Cutting). Key findings reveal: Game-Type Influence on Strategy Choice: visual approaches **dominated in geometric games (Pizza Cutting)**, numerical methods **prevailed in pattern-based games (Jumping Frogs)**. Cognitive Progression Traces: initial reliance on random trial-and-error **decreased as games progressed, evolving toward** systematic strategies. Critical insight: Consistent use of refutation across all games signals persistent epistemological vigilance. Pre-service teachers primarily used numerical and visual strategies to generalise games trial and error in numerical, and one-by-one in verbal approaches. These strategies supported or refuted predictions, effectively aiding in the development of accurate mathematical generalizations.

Findings for the Third Sub-Problem

The third sub-problem of the research is: *“What are the different forms of representation that pre-service teachers use to generalise games?”* The findings related to this question are presented in Table VIII.

As shown in Table VIII, it was observed that the pre-service teachers utilized one or, at times, multiple forms of representation, including tables, figures or diagrams, lists, numerical, verbal, and algebraic representations, when generalizing about games.

Table VIII. Representation forms used by pre-service teachers to generalise games

R	Game 1	Game 2	Game 3	Game 4
Table				
Figure or diagram				
List	3, 8, 15, 24, 35, ...	5, 13, 21, 29, 37, 45, ...	—	1, 2, 4, 7, 11, 16, ...
Numeric	11 moves for 3.2, 19 moves for 4.3, 23 moves for 3.5	5 in 2x2, 13 in 3x3, 21 in 4x4, 29 in 5x5 is the minimum number of moves	If (a,b)=1, the number of squares through which the diagonal passes a+b-1	It progresses as 1 for 0, 2 for 1, 4 for 2, 7 for 3, 11 for 4, and 16 for 5.
Verbal	If frogs of the same colour are consecutive, the game is locked	Since diagonals aren't allowed, the hypotenuse must be navigated with downward and leftward movements	When the side lengths are co-prime, we subtract one from their sum.	If we make each cut in such a way that it cuts the previous cuts, we get the maximum number of pieces
Algebraic	$mn + m + n$	$8n - 11$	$m + n - \text{GCD}(m, n)$	$\frac{n \cdot (n + 1)}{2} + 1$

Findings for the Fourth Sub-Problem

The fourth sub-problem of the research is: “*What are the issues that pre-service teachers have difficulty in generalizing?*” The findings related to the gameplay process are presented in Table IX within the framework of the actions.

Table IX. Issues that pre-service teachers have difficulty in generalizing

Actions	Difficulties	Some example cases
Understanding the game	Misunderstanding the game or violating the game rules	<i>In the diagonal game, dealing with incorrect results because the squares are not smooth (PT13)</i>
Observation	Not recording observation results, not organizing data, not knowing how to organize data, needing more observations	<i>Thinking that generalization cannot be made according to a certain number of game results (PT18)</i>
Strategy development	Not looking for a strategy for the game, not using data representations	<i>Not being able to make sense of the game, just writing the observation results (PT24)</i>
Prediction and inference	Focus on irrelevant variables	<i>In the diagonal game, it is important whether the side lengths are odd or even (PT7)</i>
Test-verify-refute	Unable to rely on few trial results and test predictions and inferences on special cases	<i>It wouldn't be $8n-3$. n starts from 1, not 2. It's not 5 for $n=2$ (PT17)</i>
Generalization	Inability to formulate observation results, inability to verbalize generalizations, inability to form a generalization covering all special cases, lack of prior knowledge	<i>It is difficult to work with letters instead of numbers (PT15). I know what the next step should be, but I don't know how to generalise it (PT26)</i>
Proof	Considering the proof or justification unnecessary, not making an effort to prove, not knowing or using proof methods	<i>What shall we prove? Isn't observation enough? (PT11). Provided for all justification is unnecessary (PT20). How will we prove it?(PT16)</i>

Table IX shows pre-service teachers struggled with identifying variables, transitioning from numerical to algebraic representations, and particularly with data organization/interpretation during generalization and proof stages.

Discussion

This study examined how pre-service mathematics teachers developed mathematical generalizations within game-based learning environments. To ensure structural alignment, the discussion is organized according to the research questions and grounded in patterns documented in the Results section.

How did pre-service teachers reach generalizations while playing the games?

Findings indicated that participants typically began with intuitive and informal conjectures based on limited numerical cases. For example, in

the Jumping Frogs Game, several participants initially proposed symmetric movement rules that later failed under asymmetric conditions, creating epistemic turning points in their reasoning. These moments of failure did not terminate inquiry; rather, they redirected attention toward structural relationships embedded in the game system.

This pattern supports Vinner's (1997) argument that intuitive reasoning can be misleading, yet the present findings extend this perspective by demonstrating how game-based environments systematically surface and challenge intuition through embodied feedback. Unlike static tasks, games frequently prompted participants to confront contradictions through rule-governed outcomes, transforming incorrect conjectures into opportunities for structural refinement. In this respect, the study extends research that treats misconceptions primarily as obstacles by showing how they can function as productive catalysts for generalization within dynamic, rule-based environments (Abrahamson et al., 2020a).

What types of generalizations and strategies did participants use?

Consistent with prior research on generalization (Becker & Rivera, 2006; Krebs, 2005; Ma, 2007), participants relied heavily on numerical experimentation and pattern tracking. However, this study reveals a distinctive game-driven mechanism: strategy refinement was often triggered by game constraints, such as parity limitations in the Sliding Puzzle or maximization demands in the Pizza Cutting Game. These constraints pushed learners beyond surface pattern recognition toward deeper structural analysis.

While previous studies document students extending patterns beyond available data (Yeşildere İmre et al., 2017), the present findings introduce an important nuance: in game contexts, the need to win functioned as a cognitive driver compelling participants to test, revise, and stabilize their conjectures. Strategic necessity, rather than teacher prompting alone, frequently propelled the shift from empirical to structural generalization. This suggests that game constraints within the context of this study play a more active epistemic role than has been emphasized in prior generalization research.

What representations did participants use in forming generalizations?

Participants used multiple representations, including lists, tables, diagrams, verbal rules, and algebraic expressions. Tables and lists were the most frequently used representations, consistent with the findings of Yıldırım and Albayrak (2016). However, unlike prior studies where representation shifts often followed teacher guidance, here transitions between representations were frequently induced by the evolving demands of gameplay.

For example, in the Diagonal Game, participants moved from grid drawings to algebraic expressions when visual counting became inefficient. This supports Tall et al.'s (2012) view that conceptual development involves bridging embodied and symbolic worlds, but the present study adds that game progression itself can compel such bridging. Participants who used multiple representations formed more stable generalizations, whereas those who avoided representational tools struggled, highlighting representation as not merely supportive but structurally necessary in game-mediated generalization processes.

What difficulties did participants experience in generalizing and proving?

A persistent difficulty emerged in the proof stage. Many participants posed foundational questions such as “What counts as proof?” and “Is testing many cases enough?” Even after producing correct generalizations, formal justification remained limited. This finding aligns with Stylianides' (2007) distinction between empirical verification and deductive proof, reinforcing the well-documented challenge of transitioning from observed regularity to formal reasoning.

However, the game context introduced an intermediate layer that is less emphasized in previous research: social validation. In several cases, generalizations were collectively accepted after shared testing and peer critique. Although this did not constitute formal proof, it reflects what Stylianides (2007) describes as social proof schemes. The study therefore suggests that games may serve as preparatory environments for proof by normalizing justification as a communal practice before it becomes a formal deductive requirement.

Broader Pedagogical Implications

Games did more than increase engagement (Shute & Ke, 2012; Rodríguez-Alveal & Koparan, 2023; Mert & Koparan, 2024; Koparan & Meletiou-Mavrotheris, 2025); they appeared to reframe how mathematical difficulty was experienced by participants. In a discipline often associated with anxiety (Foley et al., 2017), the uncertainty inherent in games (Costikyan, 2013) repositioned struggle as a natural component of exploration. Unlike static problem settings (Sinclair et al., 2016; Kafoussi & Fessakis, 2020), game dynamics embedded generalization within action, interaction, and strategic necessity.

By integrating multiple mathematical domains (Stacey, 2006) and requiring iterative refinement, games helped bridge the gap between abstraction and experience (Tall et al., 2012). These findings contribute to the broader discourse on game-based learning (Abrahamson et al.,

2020b) while responding to international concerns about mathematical reasoning and justification highlighted in large-scale assessments (OECD, 2018; Che et al., 2022).

Conclusion

This study demonstrates that game-based learning environments can function as productive contexts for developing mathematical generalization. Three mechanisms were particularly influential: embodied disconfirmation, where game outcomes challenged intuitive reasoning (Vinner, 1997), making structural analysis necessary rather than merely optional; representational fluidity, as game progression prompted shifts from visual and numerical reasoning toward symbolic generalization (Tall et al., 2012); and social validation as a bridge to proof, where collective testing supported early justification processes aligned with the social proof schemes described by Stylianides (2007).

Rather than positioning games merely as motivational tools, this study suggests that they can shape how generalization processes emerge. For teacher education, this implies that effective learning environments should not only present mathematical structures but also create conditions in which those structures must be discovered, tested, and justified through interaction.

Limitations

This study has several limitations. First, it was conducted with a single cohort of pre-service teachers in one institutional context, which limits the transferability of the findings. Second, the researcher's dual role as instructor-researcher may have influenced participants' responses, despite efforts to ensure reflexivity and data triangulation. Third, the study reflects a specific instructional period and does not address the long-term development of generalization skills. Finally, while game-based activities supported the emergence of generalizations, they did not consistently lead to formally justified statements, suggesting that additional instructional support is needed to foster deductive reasoning.

Implications and Future Research

Future research should examine how generalization processes developed in game-based settings transfer to non-game mathematical contexts and how structured proof instruction can build on exploratory phases initiated through gameplay. Comparative studies across institutions, age groups, and cultural contexts would help clarify the generalizability of these findings. Further investigation into digital adaptations of low-cost games may

also support equitable access, particularly in contexts where mathematics anxiety remains prevalent (Foley et al., 2017; Che et al., 2022).

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Bibliographic References

- Abrahamson, D., et al. (2020a). Embodied design of digital resources for mathematics education: Theory, methodology, and framework of a transdisciplinary research program. *ZDM Mathematics Education*, 52(7), 1277–1291. <https://doi.org/10.1007/s11858-020-01189-0>
- Abrahamson, D., et al. (2020b). Toward an embodied-interaction design framework for mathematical concepts. *Digital Experiences in Mathematics Education*, 6(1), 117–144. <https://doi.org/10.1007/s40751-020-00061-7>
- Becker, J. R., & Rivera, F. (2006). Sixth graders' figural and numerical strategies for generalizing patterns in algebra. In S. Alatorre, J. L. Cortina, M. Saiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 95–101). Universidad Pedagógica Nacional.
- Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review*, 31(1), 21–32.
- Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review*, 31(1), 21–32.
- Cai, Y. Y., Lu, B. F., Fan, Z. W., Indhumathi, C., Lim, K. T., Chan, C. W., Jiang, Y., & Li, L. (2006). Bio-edutainment: learning life science through X

- gaming. *Computers & Graphics*, 30(1), 3–9. <https://doi.org/10.1016/j.cag.2005.10.009>
- Carpenter, T. P., & Franke, M. L. (2001). Developing algebraic reasoning in the elementary school. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (pp. 155–162). University of Melbourne.
- Che, M., et al. (2022). Cross-cultural analysis of mathematics anxiety: Comparing East Asian and U.S. preservice teachers. *International Journal of Science and Mathematics Education*, 20(3), 561–580. <https://doi.org/10.1007/s10763-021-10176-z>
- Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–333). Routledge.
- Costikyan, G. (2013). *Uncertainty in Games*. The MIT Press.
- Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, 38(3), 194–229. <https://doi.org/10.2307/30034978>
- Ellis, A. B., Tillema, E. S., Lockwood, E., & Moore, K. C. (2012). Generalization across domains: The relating of ideas in mathematics. *For the Learning of Mathematics*, 32(3), 8–14.
- Ernest, P. (1986). Games: A rationale for their use in the teaching of mathematics in school. *Mathematics in School*, 15(1), 2–5.
- Foley, A. E., Herts, J. B., Borgonovi, F., Guerriero, S., Levine, S. C., & Beilock, S. L. (2017). The math anxiety-performance link: A global phenomenon. *Current Directions in Psychological Science*, 26(1), 52–58. <https://doi.org/10.1177/0963721416672463>
- Gerring, J. (2007). *Case study research: Principles and practices*. Cambridge University Press.
- Herbst, P., & Chazan, D. (2011). Research on teaching and learning proof: Taking stock and moving forward. In G. Hanna, M. de Villiers, & D. A. Reid (Eds.), *Proof and proving in mathematics education: The 19th ICMI Study* (pp. 407–433). Springer. https://doi.org/10.1007/978-94-007-2129-6_15
- Juul, L. (2005). *Half-real: Video games between real rules and fictional worlds*. MIT Press.
- Kafoussi, S., & Fessakis, G. (2020). Kindergarten children's generalization of patterns during play with digital games. *Early Childhood Education Journal*, 48(4), 483–494. <https://doi.org/10.1007/s10643-019-01020-9>
- Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, 8(1), 139–151.

- Koparan, T. (2019). Teaching game and simulation based probability. *International Journal of Assessment Tools in Education*, 6(2), 235-258.
- Koparan T. (2021). Yükseköğretimde dijital oyun tabanlı öğrenme ortamından yansımaların incelenmesi [Investigation of Reflections from the Digital Game-Based Learning Environment in Higher Education]. *Journal of Higher Education and Science*, 11(3), 503-515.
- Koparan, T. (2022). The Impact of a Game and Simulation Based Probability Learning Environment on the Achievement and Attitudes of Prospective Teachers, *International Journal of Mathematical Education in Science and Technology*, 53(9), 2319-2337.
- Koparan, T., & Meletiyou-Mavrotheris, M. (2025). Reflections from course design supported by games, simulation and history: the oldest probability problem, *Teaching Mathematics and its Applications: An International Journal of the IMA*, hraf006, <https://doi.org/10.1093/teamat/hraf006>
- Krebs, A. S. (2005). Studying students' reasoning. *Mathematics Teaching in the Middle School*, 10(6), 284-287.
- Lester, J. C., Spires, H. A., Nietfeld, J. L., Minogue, J., Mott, B. W., & Lobeni, E. V. (2014). Designing game-based learning environments for elementary science education: A narrative-centered learning perspective. *Information Sciences*, 264, 4-18. <https://doi.org/10.1016/j.ins.2013.09.005>
- Mason, J., Stacey, K., & Burton, L. (2010). *Thinking mathematically* (2nd edition), Edinburgh: Pearson.
- Mert, B., & Koparan, T. (2024). Matematik Öğretiminde Eğitsel Oyunların Akademik Başarıya Etkisi: Bir Meta-Analiz Çalışması [The Effect of Educational Games on Academic Achievement in Mathematics Teaching: A Meta-Analysis Study]. *Karaelmas Eğitim Bilimleri Dergisi*, 12(1), 45-60.
- Muchlisin, A., Sa'dijah, C., Parta, I. N., & Rahardi, R. (2022). Exploring students' generalization process. *Journal on Mathematics Education*, 13(3), 511-532. <https://doi.org/10.22342/jme.v13i3.pp511-532>
- National Council for Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- OECD. (2018). *The future of education and skills: Education 2030*. OECD Publishing.
- Patton, M. Q. (2015). *Qualitative Research & Evaluation Methods: Integrating Theory and Practice* (4th ed.). SAGE.
- Perrotta, C., Featherstone, G., Aston, H., & Houghton, E. (2013). *Game-based learning: Latest evidence and future directions*. National Foundation for Educational Research.
- Piaget, J. (1971). *The theory of stages in cognitive development*. McGraw-Hill.

- Polya, G. (1945). *How to solve it* (2nd ed.). Princeton University Press.
- Rodríguez-Alveal, F., & Koparan, T. (2023). Probabilistic Thinking in Mathematics Training Teachers: An Approach From Random Games. *Revista Fuentes*, 25(3), 293-304.
- Shute, V. J., & Ke, F. (2012). Games, learning, and assessment. In *Assessment in game-based learning* (pp. 43–58). Springer.
- Siegler, R. S. (2006). Microgenetic analyses of learning. In D. Kuhn & R. S. Siegler (Eds.), *Handbook of child psychology* (Vol. 2, pp. 464–510). Wiley.
- Sinclair, N., Pimm, D., & Skovsmose, O. (2013). *Mathematics education and the body*. Sense Publishers.
- Sinclair, N., et al. (2016). Digital games, design, and learning: A systematic review and meta-analysis. *Review of Educational Research*, 86(1), 79–122. <https://doi.org/10.3102/0034654315582065>
- Stacey, K. (2006). *What Is Mathematical Thinking and Why Is It Important?* University of Melbourne, Australia.
- Stake, R. E. (2006). *Multiple case study analysis*. Guilford Press.
- Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 38(3), 289–321. <https://doi.org/10.2307/30034976>
- Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. (2012). Cognitive development of proof. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education* (pp. 13–49). Springer.
- Timmermans, S., & Tavory, I. (2012). Theory construction in qualitative research: From grounded theory to abductive analysis. *Sociological Theory*, 30(3), 167-186. <https://doi.org/10.1177/0735275112457914>
- Vinner, S. (1997). From intuition to inhibition – Mathematics, education and other endangered species. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of The International Group for the Psychology of Mathematics Education* (1), 63-78.
- Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48(1), 101–119. <https://doi.org/10.1023/A:1015535623863>
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.
- Yeşildere-İmre, S., Akkoç, H., & Baştürk-Şahin, B. N. (2017). Middle school students' mathematical generalization abilities with the use of different representations. *Turkish Journal of Computer and Mathematics Education*, 8(1), 103-129.

- Yin, R. K. (2018). *Case study research and applications: Design and methods* (6. bs.). Sage.
- Yıldırım, Z., & Albayrak, M. (2016). Examining the understanding levels of secondary school students on the subject of linear relationship according to different representation styles. *Adnan Menderes University Faculty of Education Journal of Educational Sciences Journals*, 7(2), 11-26.
- Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, 60(3), 297–321. <https://doi.org/10.1007/s10649-005-1830-4>

