Report on the outcomes of a Short-Term Scientific Mission

Action number: CA22145

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STSM to Poznan, Poland, 3rd – 9th March, 2024

Details of the STSM

Title: Client-Waiter games on graphs

Start and end date: 03/03/2024 to 09/03/2024

Description of the work carried out during the STSM

The primary goal of the STSM was for the grantee to visit the Discrete Mathematics Department at Faculty of Mathematics and Computer Sciences in Poznan, Poland, for a week, and engage in a scientific collaboration on Positional Games with Małgorzata Bednarska-Bzdęga, a researcher with more than 25 years of experience in the field. The discussions attracted another two scholars based in the same institution – Sylwia Antoniuk, who has a previous experience working in positional games, and Tomasz Łuczak, one of the pioneers of the field and author of many seminal papers in the area.

Positional Games Theory deals with a subclass of combinatorial games. It aims at providing a solid mathematical footing for a variety of two-player games of perfect information, usually played on discrete objects, ranging from such popular recreational games as Tic-Tac-Toe and Hex to purely abstract games played on graphs and hypergraphs. The field has experienced tremendous growth in recent years.

In full generality, the structure of a positional game is defined by *its board,* a finite set, and *its winning sets,* a collection of subsets of the board. In the example of Tic-Tac-Toe the board is the set of nine squares, and the winning sets are the eight triples of squares. Moving on to the gameplay, there are two players who alternately claim unclaimed elements of the board until all the elements are claimed. There are several conventions for determining the winner, and the most studied one is the Maker-Breaker one: the players are named Maker and Breaker, Maker wins if she claims all elements of a winning set, and Breaker wins otherwise – if the game is finishes without a Maker's win.

In an (m:b) biased Maker-Breaker game, Maker claims m elements per move, and Breaker claims b elements per move. We attacked several exciting open problems dealing with biased Maker-Breaker games on edge sets of graphs. Our primary focus was on standard well-studied games played on edge sets of the edge set of a complete graph $E(K_n)$ – the Connectivity game, the Perfect matching game, the Hamiltonicity game as well as the *H*-game, with notable special cases like the triangle game, and more generally the cycle game. One of the central questions in the field Positional Games is to, given a game from the above set of games, determine the bias b=$b(n)$ for which a Maker's win in (1:b) biased game turns into a Breaker's win – the so-called threshold bias. The leading term is determined for all the mentioned games on graphs.

A suggestion that came from Prof. Łuczak was to look at a variant of the games with limited token supply, which was not initially planned but proved out to be a great success as a substantial progress was made in that direction. In this setting, each of the players has a limited number of moves, and after she plays all of them for all subsequent moves she must lift (and reposition) one already played move.

We first realized that, in this setting, the so-called Box game Is a useful auxiliary game to study first, as it is helpful for all the games where the graph property defining the winning sets is related to the minimum degree (like Min-degree-1, Connectivity, Perfect Matching, Hamiltonicity). From there we moved on to studying all the mentioned games with the limited token supply, relating the bias values to the threshold bias.

Description of the STSM main achievements and planned follow-up activities

We are happy to report that the STSM achieved its expected outcomes, as we have made a considerable progress in several of the questions that we addressed. A particular advance was achieved with estimating the token supply needed for either of the players to win the biased versions of the well-studied games of Box, Connectivity, Perfect matching, Hamiltonicity, H-game and several others, in situations when we already know that the player can win.

The plan of the four researchers involved is to organize a number of follow up online meetings in order to coordinate the work on the details as well as the write-up of the results, which will eventually lead to a complete publication ready for submission for possible publication in a renowned journal. On top of that, several new promising directions of research are presenting themselves, opening up a possibility for further collaboration on related follow up questions, equally for scientists that are already involved in this project and the others interested in studying games on graphs.